Definable Semigroups in o-Minimal Structures

Eduardo Magalhães
Master in mathematics
Department of Mathematics
Faculty of Sciences of the University of Porto
2025

Definable Semigroups in o-Minimal Structures

Eduardo Magalhães
Dissertation carried out as part of the Master in Mathematics
Department of Mathematics
2025

Supervisor
Prof. Mário Edmundo,
Faculty of Sciences of the University of Lisbon

Co-supervisor
Prof. Jorge Almeida,
Faculty of Sciences of the University of Porto

Acknowledgements

As everyone I want to address here are native Portuguese speakers, I will continue in Portuguese.

Em primeiro lugar, muito obrigado ao meu orientador, Professor Mário Edmundo, que esteve sempre disponível para me ajudar e cuja supervisão foi essencial para a elaboração deste projeto. Agradeço também profundamente ao meu coorientador, Professor Jorge Almeida, cuja introdução à teoria de semigrupos me foi extremamente útil ao longo desta jornada.

Um obrigado enorme aos meus pais, que me apoiaram incondicionalmente desde sempre, que me orientaram e guiaram para eu me tornar a pessoa que sou hoje. Muito obrigado aos meus irmãos, Alexandre e Leonor, pelo carinho, pela companhia e por todos os momentos de descontração que me ajudaram a manter o equilíbrio durante todo este percurso.

Agradeço imenso aos meus avós por todo o amor, e por serem sempre uma fonte de inspiração e sabedoria ao longo da minha vida.

Obrigado aos meus padrinhos, tios e primos pelas palavras de incentivo e por celebrarem comigo cada pequena conquista. Um grande beijinho à Tótota (Isaura), que me viu nascer e me acompanhou durante todas as fases da minha vida. Obrigado ao Amadeu e à Luz, que tão rapidamente me acolheram como parte da família, pelo apoio constante.

Obrigado aos meus colegas e amigos, especialmente ao Tomás Barbosa, à Maria Ana Sottomayor Barbosa, ao Diogo Santos, ao Hugo Couto, ao Martim Pinto, ao Lucas Silva e ao Pedro Lourenço, que percorreram esta caminhada comigo e me deixaram partilhar com eles o meu entusiasmo.

Por fim, mas não de todo menos importante, agradeço também à minha namorada. Obrigado, Maria, por tudo. Incentivas-me todos os dias a ser a melhor versão possível de mim mesmo, e isto não teria sido possível sem ti.

Resumo

O objetivo desta dissertação é realizar um estudo geral de semigrupos definíveis em estruturas o-minimais. Começamos com uma exposição dos conhecimentos necessários de teoria de modelos, e um desenvolvimento detalhado dos conceitos básicos de o-minimalidade, com especial ênfase em resultados conhecidos sobre grupos definíveis. O restante da dissertação é dedicado ao estudo de semigrupos definíveis. A abordagem seguida consiste em tomar resultados clássicos - quer da teoria de grupos definíveis, quer da teoria de semigrupos topológicos - e investigar se estes podem ser generalizados para o contexto dos semigrupos definíveis.

Mostramos que vários teoremas clássicos sobre semigrupos compactos permanecem válidos neste novo contexto. Concretamente, provamos a existência de idempotentes em semigrupos definivelmente compactos, a existência de ideais mínimos definíveis à esquerda e à direita, bem como a existência de um núcleo definível. Estudamos também paragrupos definivelmente compactos como uma extensão natural dos grupos definíveis, e demonstramos que muitos resultados centrais da teoria de grupos definíveis se generalizam para paragrupos se, e somente se, o paragrupo for $\mathscr{H}$-finito.

Palavras-chave: teoria de modelos, semigrupos, o-minimalidade

Abstract

The goal of this dissertation is to carry out a general study of definable semigroups in o-minimal structures. We begin with an exposition of the necessary model-theoretic background and a detailed development of the basics of o-minimality, with particular emphasis on known results concerning definable groups. The remainder of the thesis is devoted to the structure theory of definable semigroups. The guiding approach throughout is to take classical results - either from the theory of definable groups or from topological semigroup theory - and investigate whether they can be generalized to the setting of definable semigroups.

We show that several classical theorems about compact semigroups remain valid in this context. Specifically, we prove the existence of idempotents in definably compact definable semigroups, the existence of definable minimal left and right ideals, and the existence of a definable kernel. We also study definably compact paragroups as a natural extension of definable groups, and show that many key results from the theory of definable groups generalize to paragroups if and only if the paragroup is $\mathscr{H}$-finite.

Keywords: model theory, semigroups, o-minimality

Table of Contents

Introduction

Model theory is an area of mathematical logic that studies mathematical structures through the lens of first-order logic. Although many foundational results were discovered in the first half of the 20th century, the name ”Model Theory” was first coined by Alfred Tarski only in 1954 (see [1]). Almost thirty years later, in 1986, the definition of an o-minimal structure was introduced by A. Pillay and C. Steinhorn in ”Definable Sets in Ordered Structures I” [2], motivated by the work of L. Van Den Dries in [3]. At around the same time, in 1984, A. Grothendieck published the paper ”Esquisse d’un Programme” [4] where, among other things, he justifies his views for the need of a re-foundation of topology, with the goal of excluding pathological phenomena and fitting to the geometry of semialgebraic and semianalytic sets. He named this new reformulation of topology ”tame topology” (topologie modérée). In the paper, he never explicitly defined or constructed the tame topology that he envisioned and simply stated some properties that he believed such topological framework would need to have.

As it turns out, o-minimality does provides a geometric and topological framework where the typical pathologic behaviour present in classical topology and algebraic geometry do not exist. This is accomplished by focusing only on definable objects within the structure, essentially ’filtering’ out any unwanted behaviour. For example, in an o-minimal structure, definable maps are always well-behaved and definable sets can not exhibit fractal-like behaviour. Because of this, many see o-minimality as a prime candidate for the notion of tame topology envisioned by Grothendieck.

In recent decades, the theory of definable groups in o-minimal structures has evolved into a robust and elegant framework that highlights how o-minimality restricts and influences the algebraic behaviour of definable groups. This raises a natural question: to what extent can these results and techniques be extended to more general algebraic objects, such as semigroups?

The motivation of this dissertation is thus to investigate whether classical theorems concerning semigroups and topological semigroups remain valid when translated to the definable setting within o-minimal structures, and whether results about definable groups in o-minimal structure carry over to definable semigroups. In doing so, we not only aim to generalize the theory of definable groups, but also contribute to the broader program of understanding algebraic and topological behaviour under the geometric constraints of tame topology.

One of my main objectives when writing this dissertation was to make it as self-contained as possible.To that end, Chapter 1 provides a comprehensive overview of all the model-theoretic concepts and tools employed in the subsequent chapters. This includes topics such as the compactness theorem, the Löwenheim–Skolem theorem, types, saturated structures, and others. References to relevant results from Chapter 1 are given as needed throughout Chapters 2 and 3, so that any reader already familiar with model theory can skip this chapter altogether and only refer to it as necessary. The primary references for this chapter are Marker's book "Model Theory: An Introduction" [5], which was also the book I used to first learn these concepts myself, and the course notes by M. Edmundo [6].

In Chapter 2, we begin by introducing the concept of an o-minimal structure, which is central to this work. The main reference I used for this Chapter is Van Den Dries' book "Tame Topology and O-minimal Structures" [7] which, to the best of my understanding, has become a standard introduction to the subject. This chapter is divided into three main sections. The first section presents and proves the Monotonicity Theorem and the Cell Decomposition Theorem — two results that were pivotal in the early development of the theory of o-minimal structures following their introduction by A. Pillay, C. Steinhorn and J. Knight in both [2] and [8]. In the second section, we discuss some important known results about definable groups in o-minimal structures. Specifically, we introduce the t-topology, first defined by A. Pillay in [9], which has played a crucial role in the study of definable groups. Followed by some properties of the t-topology, we briefly discuss definably compactness, which was introduced by C. Steinhorn and Y. Peterzil in [10], and will play a central role in Chapter 3. Finally, the third section introduces the notions of definable spaces and definable manifolds. These concepts will serve as a foundational framework for the results presented in Chapter 3.

With this being said, the objective of Chapter 3 is to explore the possibility of generalizing the properties of definable groups in o-minimal structures to the broader context of definable semigroups. For that, much like in Chapter 1, we start with a brief exposition of basic definitions and facts from semigroup theory. Starting from Section 3.2, the content presented is my original contribution to the subject. I start by generalizing to the setting of definable spaces some technical results concerning definable compactness, originally established by Y. Peterzil and A. Pillay in [11] and further generalized by M. Edmundo and G. Terzo in [12]. Using these tools, we proceed to analyse the algebraic structure of definable semigroups equipped with a compatible definably compact definable space structure. In particular, we prove the existence of idempotents, the existence of a unique minimal definable ideal, and explore several corollaries that follow from these results.

1. Model Theory

Mathematical structures often exhibit rich logical properties that can be systematically analysed through the lens of model theory. In this chapter, we will explore some fundamental albeit essential concepts from model theory that will lay the groundwork for the rest of this work.

1.1. Foundational Concepts

1.1.1 Signatures and Structures

One of the basic objects we study and work with in model theory are structures, and as such, we begin by defining what a structure is.

Definition 1.1.1. A *first-order signature*, or just signature, is given by a tuple $\sigma = (I, J, K, \rho)$ where I, J, K are disjoint sets and $\rho : I \cup J \to \mathbb{N}$ is a function. The number assigned to every element of I and J by ρ is called its arity.

Definition 1.1.2. Given a signature $\sigma = (I, J, K, ar)$, a σ-*structure* is a tuple

$$\mathcal{M} = (M, (R_i)_{i \in I}, (f_j)_{j \in J}, (c_k)_{k \in K})$$

where:

- M is a non-empty set called the universe or domain of $\mathcal{M}$, denoted by $dom(M)$;
- for each $i \in I$, $R_i \subseteq M^n$ is an n-ary relation on M, with $n = \rho(i)$;
- for each $j \in J$, $f_j : M^n \to M$ is an n-ary function on M, with $n = \rho(j)$;
- for each $k \in K$, $c_k \in M$ is a constant of M.

Example 1.1.3. *Let $I = \emptyset$, $J = \{a, b\}$ and $K = \{c, d\}$ with arity function given by $\rho(a) = 1$ and $\rho(b) = 2$. Consider the signature $\sigma = (I, J, K, \rho)$. The following are some examples of σ-structures:*

- $(\mathbb{R}, exp, \cdot, 0, 1)$
- $(\mathbb{Z}, |\ |, +, 2, 3)$
- $(\mathcal{P}(\mathbb{N}), {}^c, \cap, \emptyset, \mathbb{N})$

Before continuing any further, I will define the familiar notions of homomorphism, isomorphism, and embedding for general σ-structures:

Definition 1.1.4. Let

$$\mathcal{A} = (A, (R_i)_{i\in I}, (f_j)_{j\in J}, (c_k)_{k\in K})$$

and

$$\mathcal{B} = (A, (S_i)_{i\in I}, (g_j)_{j\in J}, (d_k)_{k\in K})$$

be two σ-structures. A map $\pi : A \to B$ is a *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if the following are true:

- For each $i \in I$ and $a_1, \dots, a_{\rho(i)} \in A$, if $(a_1, \dots, a_{\rho(i)}) \in R_i$ then $(\pi(a_1), \dots, \pi(a_{\rho(i)})) \in S_i$;
- For each $j \in J$ and $a_1, \dots, a_{\rho(j)} \in A$, $\pi(f_j(a_1, \dots, a_{\rho(j)})) = g_j(\pi(a_1), \dots, \pi(a_{\rho(j)}))$;
- For each $k \in K$, $\pi(c_k) = d_k$.

We say that a homomorphism $\pi : A \to B$ is an *embedding* if:

- π is injective;
- For each $i \in I$ and $a_1, \dots, a_{\rho(i)} \in A$, then $(a_1, \dots, a_{\rho(i)}) \in R_i$ if and only if $(\pi(a_1), \dots, \pi(a_{\rho(i)})) \in S_i$.

By an *isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ we mean a bijective embedding from $\mathcal{A}$ to $\mathcal{B}$. If there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, we say that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* and denote it by $\mathcal{A} \simeq \mathcal{B}$.

If $A \subseteq B$, when we say that $\mathcal{A}$ is a *substructure* of $\mathcal{B}$ or that $\mathcal{B}$ is an *extension* of $\mathcal{A}$, denoted by $\mathcal{A} \leq \mathcal{B}$, if the inclusion map $\iota : A \hookrightarrow B$ is an embedding.

Example 1.1.5.

1. *Let $\mathcal{A} = (\mathbb{N}, |)$ where $|$ is a binary relation given by $n \mid m$ if and only if n divides m, and let $\mathcal{B} = (\mathbb{N}, \leq)$, where $\leq$ is the usual order of $\mathbb{N}$. Then the identity map $\iota : \mathcal{A} \to \mathcal{B}$ is an injective homomorphism, as $n \mid m$ implies that $n \leq m$. However, it is not an embedding because, for example, $2 \leq 3$ but $2 \nmid 3$.*

2. *Let $\mathcal{A} = (\mathbb{Q}, <, +, \cdot, 0, 1)$ and $\mathcal{B} = (\mathbb{R}, <, +, \cdot, 0, 1)$. Then the identity map $\iota : \mathcal{A} \to \mathcal{B}$ is an embedding.*

3. *Let* $\mathcal{A} = (\mathbb{R}, <, +, 0)$ *and* $\mathcal{B} = (\mathbb{R}^{>0}, <, \cdot, 1)$. *Then the map* $f : \mathcal{A} \to \mathcal{B}$ *given by* $f(x) = e^x$ *is an isomorphism.*

Before continuing, note that, using the notation from Definition 1.1.4, $\mathcal{A}$ is a substructure of $\mathcal{B}$ if and only if $A \subseteq B$ and:

- For each $i \in I$, $R_i = A^{\rho(i)} \cap S_i$
- For each $j \in J$, $f_j = g_j|_{A^{\rho(j)}}$
- For each $k \in K$, $c_k = d_k$ and in particular, $d_k \in A$

This implies that, given a σ-structure

$$\mathcal{B} = (A, (S_i)_{i \in I}, (g_j)_{j \in J}, (d_k)_{k \in K})$$

and a subset $A \subseteq B$ such that:

- for each $j \in J$, g_j is closed in A
- for each $k \in K$, $d_k \in A$

we can construct a substructure $\mathcal{A} = (A, (R_i)_{i \in I}, (f_j)_{j \in J}, (c_k)_{k \in K})$ with universe A by taking

- $R_i := A^{\rho(i)} \cap S_i$
- $f_j := g_j|_{A^{\rho(j)}}$
- $c_k := d_k$

1.1.2 Languages and Formulas

Another very important object we work with and study in model theory are theories. Informally speaking, a theory is simply a set of ”properties” that some structures of interest have, and those ”properties” are encoded via the use of formulas. Although we will not define right now formally what a theory is, we will start by defining it is fundamental building block: formulas. To do so, we start by the preliminary definition of a first-order language.

Definition 1.1.6. Let $\sigma = (I, J, K, \rho)$ be a signature. The *first-order language* with signature σ, denoted by $\mathcal{L}_\sigma$ is a formal language consisting of:

Logical Symbols:

- A countably infinite number of variables $v_1, v_2, \ldots$;
- The equality symbol $=$;
- The logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$;
- The universal and existential quantifiers $\forall, \exists$;
- Parenthesis (,) and a comma , .

Non-logical symbols:

- For each $i \in I$, an $\rho(i)$-ary relational symbol P_i;
- For each $j \in J$, an $\rho(j)$-ary functional symbol F_j;
- For each $k \in K$, a constant symbol c_k.

When the signature is clear from the context or when it is arbitrary, we simply write $\mathcal{L}$ instead of $\mathcal{L}_\sigma$. We define $|\mathcal{L}_\sigma| = \max\{|I|, |J|, |K|\}$, and we say that the language is finite [countable] if $|\mathcal{L}_\sigma|$ is finite [countable].

Note that, to specify a first-order language, we only need to know its non-logical symbols. Furthermore, up to renaming of non-logical symbols, there is a unique first-order language of a given signature, which gives rise to a natural correspondence between first-order languages and signatures. In fact, it is common to implicitly define a signature by giving a first-order language, and in particular if $\mathcal{L}$ is a language with signature σ, it is common to write $\mathcal{L}$-structure instead of σ-structure.

Example 1.1.7. *Some examples of languages include:*

- *The language of rings $\mathcal{L}_r = \{+, -, \cdot, 0, 1\}$ where $+, -, \cdot$ are binary functional symbols and 0 and 1 are constants;*
- *The language of ordered rings $\mathcal{L}_{or} = \mathcal{L}_r \cup \{<\}$ where $<$ is a binary relational symbol;*
- *The language of groups: $\mathcal{L}_g = \{\cdot, {}^{-1}, e\}$, where e is a constant, $\cdot$ is a binary functional symbol and ${}^{-1}$ is a unary functional symbol;*
- *The language of pure sets $\mathcal{L} = \emptyset$;*
- *The language of graphs $\mathcal{L} = \{R\}$ where R is a binary relational symbol.*

Let σ be a signature, $\mathcal{L}_\sigma$ be a language of signature σ and let $\mathcal{M}$ be an $\mathcal{L}_\sigma$-structure given by $(M, (R_i)_{i\in I}, (f_j)_{j\in J}, (e_k)_{k\in K})$. For any $i \in I$, we say that R_i is the interpretation of the relational symbol P_i of $\mathcal{L}_\sigma$, and it is common and sometimes useful to write $P_i^{\mathcal{M}}$ instead of R_i. Analogously, we sometimes write $F_j^{\mathcal{M}}$ instead of f_j and $c_k^{\mathcal{M}}$ instead of e_k for the interpretations of functional and constant symbols in $\mathcal{M}$ respectively.

Now, let $\mathcal{L}$ be an arbitrary first-order language. As with any formal language, an $\mathcal{L}$-word is simply a sequence of symbols of $\mathcal{L}$. The main goal is to use $\mathcal{L}$-words to describe properties of $\mathcal{L}$-structures. Such $\mathcal{L}$-words are called formulas. We start by defining an elementary building block in such formulas known as terms.

Definition 1.1.8. Let $\mathcal{L}$ be a first-order language with signature $\sigma = (I, J, K, \rho)$. An *$\mathcal{L}$-term* is a $\mathcal{L}$-word defined by the following set of rules:

- For each $p \geq 1$, v_p is an $\mathcal{L}$-term;
- For each $k \in K$, c_k is an $\mathcal{L}$-term;
- For each $j \in J$, if $t_1, \ldots, t_{\rho(j)}$ are $\mathcal{L}$-terms, then $F_j(t_1, \ldots, t_{\rho(j)})$ is an $\mathcal{L}$-term;
- Any $\mathcal{L}$-term is obtained from the application of the rules above a finite number of times.

Example 1.1.9. *Consider the language of rings $\mathcal{L}_r$ in example 1.1.7. Some examples of $\mathcal{L}_r$-terms include: $\cdot(v_1, -(v_3, 1))$ and $+(1, +(1, +(1, 1)))$.*

When writing terms with binary functional symbols that are meant to represent operations, for example with $+$ or $\cdot$ in the language of rings, it is common to write $t_1 + t_2$ instead of $+(t_1, t_2)$. So for the examples given in example 1.1.9, we would write those terms as $v_1 \cdot (v_3 - 1)$ and $1 + (1 + (1 + 1))$ instead.

Having defined $\mathcal{L}$-terms, we are ready to define what an $\mathcal{L}$-formula is.

Definition 1.1.10. An *atomic $\mathcal{L}$-formula* is either:

- $t_1 = t_2$ where t_1, t_2 are $\mathcal{L}$-terms;
- $R_i(t_1, \ldots, t_{\rho(i)})$, for some relational symbol R_i and terms $t_1, \ldots, t_{\rho(i)}$.

We define *$\mathcal{L}$-formulas* as follows:

- Every atomic $\mathcal{L}$-formula is an $\mathcal{L}$-formula;
- If ϕ is an $\mathcal{L}$-formula, then so is $\neg\phi$;
- If ϕ, ψ are $\mathcal{L}$-formulas, then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$ and $\phi \leftrightarrow \psi$;
- If ϕ is an $\mathcal{L}$-formula, then for any $p > 0$, so are $\forall v_p \phi$ and $\exists v_p \phi$;
- Any $\mathcal{L}$-formula is obtained from the application of the rules above a finite number of times.

Example 1.1.11. *Some examples of $\mathcal{L}_{or}$-formulas include:*

- $\forall v_1 (v_1 \cdot v_1 < 0)$
- $v_1 > 0 \leftrightarrow \exists v_2 (v_2 \cdot v_2 = 0)$
- $v_1 = 0 \vee 0 < v_1$

Another important concept is that of a free variable. Intuitively, a free variable is one that is not "bound" by any quantifier. For example, in the language of rings, consider the formula: $\forall v_1 (v_1 + v_2 = 0)$. Here v_2 is a free variable while v_1 is not, as it is "bound" by the universal quantifier $\forall v_1$.

Definition 1.1.12. We say that a *variable v_p is free* in a formula ϕ, if one of the following is true:

- ϕ is an atomic formula and v_p occurs in ϕ;
- $\phi = \neg\psi$, with v_p free in ψ;
- ϕ is of the form $\psi \wedge \chi$, $\psi \vee \chi$, $\psi \rightarrow \chi$ or $\psi \leftrightarrow \chi$, with v_p free in ψ or χ;
- ψ is of the form $\forall v_q \psi$ or $\exists v_q \psi$, with v_p free in ψ and $q \neq p$.

A formula with no free variables is called a sentence. These types of formulas play a key role in model theory, as we will see later. Examples of sentences in the language of ordered rings include: $\forall v_1 \forall v_2 (v_1 + v_2 > v_1)$, $\forall v_1 \exists v_2 (v_1 \cdot v_2 = 1)$ and $\neg(1 + 1 + 1 = 0)$.

For any set A, let A^ω denote the set $\{(a_1, a_2, \ldots) : a_n \in A \text{ for all } n \geq 1\}$ of infinite sequences of elements of A.

Given a language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathcal{M} = (M, \{R_i\}_{i\in I}, \{f_j\}_{j\in J}, \{e_k\}_{k\in K})$ and a term t, we can define a function

$$t^{\mathcal{M}} : M^{\omega} \to M$$

as follows:

for any $\bar{a} \in M^{\omega}$,

- if $t = v_p$, for some $p \geq 1$, then $t^{\mathcal{M}}(\bar{a}) = a_p$;
- if $t = c_k$, then $t^{\mathcal{M}}(\bar{a}) = e_k$;
- if $t = F_j(t_1, \ldots, t_{\rho(j)})$, then $t^{\mathcal{M}}(\bar{a}) = f_j(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{\rho(j)}^{\mathcal{M}}(\bar{a}))$.

We say that the function $t^{\mathcal{M}}$ is the *interpretation of the term* t in $\mathcal{M}$. We can think of an element $\bar{a} \in M^{\omega}$ as assigning values in M to each variable $v_1, v_2, \ldots$, and then $t^{\mathcal{M}}(\bar{a})$ simply ”evaluates” the term t when we substitute each variable by the value assigned by $\bar{a}$.

Example 1.1.13. *Consider the language $\mathcal{L} = (F, G, c)$ where F is a unary function symbol, G is a binary function symbol and c is a constant symbol and consider the term*

$$t = G(v_3, F(G(v_6, c)))$$

Then:

- *If $\mathcal{A} = (\mathbb{R}, exp, +, 1)$, then $t^{\mathcal{A}}(\bar{r}) = r_3 + exp(r_6 + 1)$*
- *If $\mathcal{B} = (\mathcal{P}(\mathbb{N}), {}^c, \cap, \mathbb{N})$, then $t^{\mathcal{B}}(\bar{S}) = S_3 \cap S_6^c$*
- *If $\mathcal{C} = (M_n(\mathbb{C}), {}^T, \cdot, I_n)$, then $t^{\mathcal{A}}(\bar{M}) = M_3 \cdot M_6^T$*

As we can see from this example, the interpretation of t only seems to depend on the variables that occur in t, and as we will see now, this is precisely the case.

Proposition 1.1.14. *Let $\mathcal{L}$ be a first-order language, $\mathcal{M} = (M, \{R_i\}_{i\in I}, \{f_j\}_{j\in J}, \{e_k\}_{k\in K})$ an $\mathcal{L}$-structure and t a $\mathcal{L}$-term. Let $\bar{a}, \bar{b} \in M^{\omega}$ such that $a_p = b_p$ for all $p \in \mathbb{N}$ such that v_p occurs in t. Then $t^{\mathcal{M}}(\bar{a}) = t^{\mathcal{M}}(\bar{b})$.*

Proof. We use induction on the terms. If $t = v_p$ for some p, then $t^{\mathcal{M}}(\bar{a}) = a_p = b_p = t^{\mathcal{M}}(\bar{b})$. If $t = c_k$, for some $k \in K$, then $t^{\mathcal{M}}(\bar{a}) = e_k = t^{\mathcal{M}}(\bar{b})$. If $t = F_j(t_1, \ldots, t_{\rho(j)})$ for some

$j \in J$ and terms $t_1, \ldots, t_{\rho(j)}$, then:

$$\begin{aligned} t^{\mathcal{M}}(\bar{a}) &= f_j(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{\rho(j)}^{\mathcal{M}}(\bar{a})) \\ &= f_j(t_1^{\mathcal{M}}(\bar{b}), \ldots, t_{\rho(j)}^{\mathcal{M}}(\bar{b})) \\ &= t^{\mathcal{M}}(\bar{b}) \end{aligned}$$

using the induction hypothesis that $t_1^{\mathcal{M}}(\bar{a}) = t_1^{\mathcal{M}}(\bar{b}), \ldots, t_{\rho(j)}^{\mathcal{M}}(\bar{a}) = t_{\rho(j)}^{\mathcal{M}}(\bar{b})$. □

Given an $\mathcal{L}$-term t, we write $t(v_1, \ldots, v_n)$ to denote that the variables that occur in t are among $v_1, \ldots, v_n$. Proposition 1.1.14 justifies the following notation: given an $\mathcal{L}$-structure and $\bar{a} \in M^\omega$, we write $t^{\mathcal{M}}(a_1, \ldots, a_n)$ instead of $t^{\mathcal{M}}(\bar{a})$, where the variables that occur in t are among $v_1, \ldots, v_n$.

Before continuing, I will introduce the following notation: for $\bar{a} \in A^\omega$, $b \in A$ and $p \geq 1$, we use $\bar{a}(p/b)$ to denote the element $(a_1, \ldots, a_{p-1}, b, a_{p+1}, \ldots) \in A^\omega$, i.e. we replace the p-th element of $\bar{a}$ by b.

We have already defined what a formula in any given language is, and as I hinted at earlier, we will use formulas to express properties of structures. Therefore, the next natural step is to define what is means for a formula to be true in some structure. The following definition is also known as Tarski's definition of truth:

Definition 1.1.15. Let $\mathcal{L}$ be a first-order language, $\mathcal{M} = (M, \{R_i\}_{i \in I}, \{f_j\}_{j \in J}, \{e_k\}_{k \in K})$ an $\mathcal{L}$-structure, ϕ an $\mathcal{L}$-formula and $\bar{a} \in M^\omega$. We define inductively what it means for ϕ to be true in $\mathcal{M}$ when we assign the values of $\bar{a}$ to the variables $v_1, v_2, \ldots$, denoted by $\mathcal{M} \models \phi[\bar{a}]$:

- If ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$;
- If ϕ is $P_i(t_1, \ldots, t_{\rho(i)})$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{\rho(i)}^{\mathcal{M}}(\bar{a})) \in R_i$;
- If ϕ is $\neg\psi$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M} \models \psi[\bar{a}]$ is not true;
- If ϕ is $\psi \wedge \chi$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M} \models \psi[\bar{a}]$ and $\mathcal{M} \models \chi[\bar{a}]$;
- If ϕ is $\psi \vee \chi$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M} \models \psi[\bar{a}]$ or $\mathcal{M} \models \chi[\bar{a}]$;
- If ϕ is $\psi \to \chi$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M} \models \psi[\bar{a}]$ implies $\mathcal{M} \models \chi[\bar{a}]$;
- If ϕ is $\psi \leftrightarrow \chi$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if, $\mathcal{M} \models \psi[\bar{a}]$ is equivalent to $\mathcal{M} \models \chi[\bar{a}]$;
- If ϕ is $\exists v_p \psi$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if there is $b \in M$ such that $\mathcal{M} \models \psi[\bar{a}(p/b)]$;

- If ϕ is $\forall v_p \psi$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if for all $b \in M$ we have $\mathcal{M} \models \psi[\bar{a}(p/b)]$.

Example 1.1.16. *Consider the language of rings and the formula $\phi = \exists v_2 (v_2 \cdot v_1 = 1)$. In $\mathcal{R} = (\mathbb{R}, +, -\cdot, 0, 1)$, we have that $\mathcal{R} \models \phi[\bar{x}]$ if and only if $x_1 \neq 0$. If $\mathcal{Z} = (\mathbb{Z}, +, -, \cdot, 0, 1)$, then $\mathcal{Z} \models \phi[\bar{k}]$ if and only if $k_1 = \pm 1$.*

Just as with Proposition 1.1.14, we will now see that the truth of a formula only depends on the values we assign to its free variables.

Proposition 1.1.17. *Let $\mathcal{L}$ be a first-order language, $\mathcal{M} = (M, \{R_i\}_{i \in I}, \{f_j\}_{j \in J}, \{e_k\}_{k \in K})$ an $\mathcal{L}$-structure and ϕ a formula. Let $\bar{a}, \bar{b} \in M^\omega$ such that $a_q = b_q$ for all $q \in \mathbb{N}$ such that v_q is a free variable in ϕ. Then $\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M} \models \phi[\bar{b}]$.*

Proof. Using induction over the definition of a formula:

If ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$. Every variable that occurs in ϕ is free, so by Proposition 1.1.14 we have that $t_1^{\mathcal{M}}(\bar{a}) = t_1^{\mathcal{M}}(\bar{b})$ and $t_2^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{b})$. Thus $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$ if and only if $t_1^{\mathcal{M}}(\bar{b}) = t_2^{\mathcal{M}}(\bar{b})$ which is equivalent to $\mathcal{M} \models \phi[\bar{b}]$.

If ϕ is $P_i(t_1, \ldots, t_{\rho(i)})$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{\rho(i)}^{\mathcal{M}}(\bar{a})) \in R_i$ if and only if $(t_1^{\mathcal{M}}(\bar{b}), \ldots, t_{\rho(i)}^{\mathcal{M}}(\bar{b})) \in R_i$ which is equivalent to $\mathcal{M} \models \phi[\bar{b}]$.

If ϕ is $\neg\psi$ then ϕ and ψ have the same free variables, and therefore by induction we have that $\mathcal{M} \models \psi[\bar{a}]$ if and only if $\mathcal{M} \models \psi[\bar{b}]$, ergo $\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M} \models \phi[\bar{b}]$. When ϕ is either $\psi \wedge \chi$, $\psi \vee \chi$, $\psi \to \chi$ or $\psi \leftrightarrow \chi$ the proof is analogous.

If ϕ is $\exists v_p \psi$, then $\mathcal{M} \models \phi[\bar{a}]$ if and only if there exists $d \in M$ such that $\mathcal{M} \models \psi[\bar{a}(p/d)]$. Consider the variable assignment given by $\bar{b}(p/d)$. Then they coincide in v_p and in any other free variable of ϕ. However, we know that $\{\text{free variables in } \psi\} \subseteq \{p\} \cup \{\text{free variables in } \phi\}$, so $\bar{a}(p/d)$ and $\bar{b}(p/d)$ coincide in every free variable of ψ and by the induction hypothesis we conclude that $\mathcal{M} \models \psi[\bar{a}(p/d)]$ if and only if $\mathcal{M} \models \psi[\bar{b}(p/d)]$, implying that $\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M} \models \phi[\bar{b}]$. In the case where ϕ is $\forall v_p \psi$, the proof is analogous. $\square$

As sentences, by definition, do not have any free variables, we immediately conclude the following:

Corollary 1.1.18. *Let $\mathcal{L}$ be a first-order language, $\mathcal{M}$ an $\mathcal{L}$-structure and ϕ an $\mathcal{L}$-sentence. Then exactly one of the following is true:*

1. *$\mathcal{M} \models \phi[\bar{a}]$ for every $\bar{a} \in M^\omega$;*

2. *$\mathcal{M} \not\models \phi[\bar{a}]$ for every $\bar{a} \in M^\omega$;*

In the first case, we use the notation $\mathcal{M} \models \phi$ and in the second case we use $\mathcal{M} \not\models \phi$.

Proposition 1.1.17 also justifies the following notation: If ϕ is an $\mathcal{L}$-formula with free variables from $v_1, \ldots, v_n$, then we write $\phi(v_1, \ldots, v_n)$. Given an $\mathcal{L}$-structure $\mathcal{M}$ and $a_1, \ldots, a_n \in M$, we write $\mathcal{M} \models \phi[a_1, \ldots, a_n]$ to denote $\mathcal{M} \models \phi[\bar{b}]$, for any $\bar{b} \in M^\omega$ such that $b_i = a_i$, for $i \leq n$.

1.1.3 Normal Forms

This subsection is more technical in nature compared to the previous ones. When proving statements about first-order formulas, it is often useful to put them into ”standardized” forms, known as normal forms. Later, as we use normal forms to prove more statements about first-order structures and formulas, the usefulness of normal forms will become apparent.

The first important thing we need to define is what means for two formulas over the same language to be equivalent.

Definition 1.1.19. We say that two $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_n)$ and $\psi(v_1, \ldots, v_n)$ are *equivalent*, if for all $\mathcal{L}$-structures $\mathcal{M}$ and for all $\bar{a} \in M^n$ we have that $\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M} \models \psi[\bar{a}]$. This is equivalent to saying that for every $\mathcal{L}$-structure $\mathcal{M}$, we have that $\mathcal{M} \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.

Note that being equivalent is an equivalence relation in the set of all $\mathcal{L}$-formulas. Some important examples of this include:

Example 1.1.20.

- *$\neg\neg\phi$ is equivalent to ϕ;*
- *$\phi \vee \psi$ is equivalent to $\neg(\neg\phi \wedge \neg\psi)$;*
- *$\phi \rightarrow \psi$ is equivalent to $\neg\phi \vee \psi$;*
- *$\phi \leftrightarrow \psi$ is equivalent to $(\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$;*
- *$\forall x\phi$ is equivalent to $\neg\exists x\neg\phi$.*

Note that, in particular, any formula is equivalent to one that just uses the logical symbols $\neg, \wedge$ and $\exists$. This means that when doing induction over the complexity of a formula, after checking that the claim is true for atomic formulas, we just need to check the case where the formula is $\neg\psi$, $\psi \wedge \chi$ and $\exists x\psi$.

Additionally, the next set of examples of equivalences of formulas are also very important and sometimes used implicitly.

Example 1.1.21.

- *$\phi \wedge \exists x\psi$ is equivalent to $\exists x(\phi \wedge \psi)$ if x does not occur free in ϕ;*
- *$\exists x(\phi \vee \psi)$ is equivalent to $\exists x\phi \vee \exists x\psi$;*
- *$\exists x\phi$ is equivalent to ϕ if x does not occur free in ϕ*

We say that a formula is a literal if it is either an atomic formula or the negation of an atomic formula, and we say that a formula is quantifier-free if it does not have any quantifiers.

Definition 1.1.22.

1. A quantifier-free formula ϕ is in *conjunctive normal form* (CNF) if it is the conjunction of one or more disjunctions of one or more literals.
2. A quantifier-free formula ϕ is in *disjunctive normal form* (DNF) if it is the disjunction of one or more conjunctions of one or more literals.

Example 1.1.23. *Let ϕ_1, ϕ_2, ϕ_3 be atomic formulas.*

- *The following are examples of formulas in CNF:*

$$\phi_1, \neg\phi_1, \phi_1 \vee \phi_2, (\neg\phi_1 \vee \phi_2) \wedge (\neg\phi_3), (\neg\phi_1 \vee \neg\phi_2) \wedge (\neg\phi_1 \vee \phi_2 \vee \phi_3)$$

- *The following are examples of formulas in DNF:*

$$\phi_1, \neg\phi_1, \phi_1 \wedge \phi_2, (\neg\phi_1 \wedge \phi_2) \vee (\neg\phi_3), (\neg\phi_1 \wedge \neg\phi_2) \vee (\neg\phi_1 \wedge \phi_2 \wedge \phi_3)$$

The following is easily proved by induction on formulas:

Proposition 1.1.24. *Any quantifier-free formula is equivalent to one in CNF and one in DNF.*

Another important form that formulas can take is the prenex normal form.

Definition 1.1.25. Let ϕ be a formula. We say that ϕ is in *prenex normal form* if it is

$$Q_i x_i \dots Q_n x_n \psi$$

where $Q_i \in \{\forall, \exists\}$ and ψ is a quantifier-free formula.

Again, by induction on the complexity of formulas, the following is easy to prove.

Proposition 1.1.26. *Any formula is equivalent to one in prenex normal form.*

1.1.4 Elementary Equivalence

We have already defined what it means for two structures to be isomorphic. Intuitively, two structures being isomorphic means that they are basically the same structure, only with different "labels" for its elements. Two structures being isomorphic is a very strong statement, which is why in the context of first-order logic, it is often useful to consider a weaker version of this, namely that of elementary equivalence.

Definition 1.1.27. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. Then we say that $\mathcal{M}$ and $\mathcal{N}$ are *elementary equivalent*, denoted by $\mathcal{N} \equiv \mathcal{M}$ if, for every $\mathcal{L}$-sentence ϕ, we have

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$$

Do note that $\equiv$ is an equivalence relation of $\mathcal{L}$-structures.

Proposition 1.1.28. *Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. If $\mathcal{M} \simeq \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.*

Proof. Let $f : \mathcal{M} \to \mathcal{N}$ be an isomorphism. We will actually show something stronger than $\mathcal{M} \equiv \mathcal{N}$, namely that for any $\mathcal{L}$-formula ϕ with free variables from $v_1, \dots, v_n$, and for all $(a_1, \dots, a_n) \in M^n$ we have

$$\mathcal{M} \models \phi[a_1, \dots, a_n] \Leftrightarrow \mathcal{N} \models \phi[f(a_1), \dots, f(a_n)]$$

Let t be an $\mathcal{L}$-term with variables from $v_1, \dots, v_n$, let $\bar{a} = (a_1, \dots, a_n) \in M^n$ and let $f(\bar{a}) = (f(a_1), \dots, f(a_n)) \in N^n$. We will start by proving that $f(t^{\mathcal{M}}(\bar{a})) = t^{\mathcal{N}}(f(\bar{a}))$ using induction.

1) If $t = v_p$, then $f(t^{\mathcal{M}}(\bar{a})) = f(a_p) = t^{\mathcal{N}}(f(\bar{a}))$;

2) If $t = c$, then

$$\begin{aligned} f(t^{\mathcal{M}}(\bar{a})) &= f(c^{\mathcal{M}}) \\ &= c^{\mathcal{N}} && (f \text{ is an isomorphism}) \\ &= t^{\mathcal{N}}(f(\bar{a})) \end{aligned}$$

3) If $t = g(t_1, \ldots, t_m)$, then

$$\begin{aligned} f(t^{\mathcal{M}}(\bar{a})) &= f(g^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_m^{\mathcal{M}}(\bar{a}))) \\ &= g^{\mathcal{N}}(f(t_1^{\mathcal{M}}(\bar{a})), \ldots, f(t_m^{\mathcal{M}}(\bar{a})))) \\ &= g^{\mathcal{N}}(t_1^{\mathcal{N}}(f(\bar{a})), \ldots, t_m^{\mathcal{N}}(f(\bar{a})))) && (\text{Induction Hypothesis}) \\ &= t^{\mathcal{N}}(f(\bar{a})) \end{aligned}$$

Now, we will use induction on formulas to prove the rest of the proposition. Let ϕ be an $\mathcal{L}$-formula.

1) If ϕ is $t_1 = t_2$, then

$$\begin{aligned} \mathcal{M} \models \phi[\bar{a}] &\Leftrightarrow t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a}) \\ &\Leftrightarrow f(t_1^{\mathcal{M}}(\bar{a})) = f(t_2^{\mathcal{M}}(\bar{a})) && (f \text{ is bijective}) \\ &\Leftrightarrow t_1^{\mathcal{N}}(f(\bar{a})) = t_2^{\mathcal{N}}(f(\bar{a})) \\ &\Leftrightarrow \mathcal{N} \models \phi[f(\bar{a})] \end{aligned}$$

2) If ϕ is $R(t_1, \ldots, t_m)$, then

$$\begin{aligned} \mathcal{M} \models \phi[\bar{a}] &\Leftrightarrow (t_1^{\mathcal{M}}(\bar{a}), \ldots, t_m^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}} \\ &\Leftrightarrow (f(t_1^{\mathcal{M}}(\bar{a})), \ldots, f(t_m^{\mathcal{M}}(\bar{a}))) \in R^{\mathcal{N}} && (f \text{ is an embedding}) \\ &\Leftrightarrow (t_1^{\mathcal{N}}(f(\bar{a})), \ldots, t_m^{\mathcal{N}}(f(\bar{a}))) \in R^{\mathcal{N}} \\ &\Leftrightarrow \mathcal{N} \models \phi[f(\bar{a})] \end{aligned}$$

3) If ϕ is $\neg\psi$, for some ψ, then

$$\begin{aligned} \mathcal{M} \models \phi[\bar{a}] &\Leftrightarrow \mathcal{M} \not\models \psi[\bar{a}] \\ &\Leftrightarrow \mathcal{N} \not\models \psi[f(\bar{a})] && (\text{Induction Hypothesis}) \\ &\Leftrightarrow \mathcal{N} \models \phi[f(\bar{a})] \end{aligned}$$

4) If ϕ is $\psi \wedge \chi$, for some ψ and χ, then

$$\begin{aligned}\mathcal{M} \models \phi[\bar{a}] &\Leftrightarrow \mathcal{M} \models \psi[\bar{a}] \text{ and } \mathcal{M} \models \chi[\bar{a}] \\ &\Leftrightarrow \mathcal{N} \models \psi[f(\bar{a})] \text{ and } \mathcal{N} \models \chi[f(\bar{a})] && \text{(Induction Hypothesis)} \\ &\Leftrightarrow \mathcal{N} \models \phi[f(\bar{a})]\end{aligned}$$

5) If ϕ is $\exists w \psi(v_1, \ldots, v_n, w)$, for some ψ, then

$$\begin{aligned}\mathcal{M} \models \phi[\bar{a}] &\Leftrightarrow \mathcal{M} \models \psi[\bar{a}, b_0] \text{ for some } b_0 \\ &\Rightarrow \mathcal{N} \models \psi[f(\bar{a}), f(b_0)] \text{ for some } b_0 && \text{(Induction Hypothesis)} \\ &\Rightarrow \mathcal{N} \models \phi[f(\bar{a})]\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{N} \models \phi[f(\bar{a})] &\Leftrightarrow \mathcal{N} \models \psi[f(\bar{a}), b_0] \text{ for some } b_0 \\ &\Rightarrow \mathcal{N} \models \psi[f(\bar{a}), f(c_0)] \text{ for some } c_0 && (f \text{ is surjective}) \\ &\Rightarrow \mathcal{M} \models \psi[\bar{a}, c_0] \text{ for some } c_0 && \text{(Induction Hypothesis)} \\ &\Rightarrow \mathcal{M} \models \phi[\bar{a}] && \square\end{aligned}$$

Despite the fact that isomorphic structures are elementary equivalent, the converse is not true in general. This is mostly due to the fact that first-order logic is not capable of describing and differentiating between different infinite cardinalities.

For example, as we will see later, any two dense linear ordered sets without endpoints are elementary equivalent. So for instance, in the language with just one binary relational symbol $<$, the structures $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$, with $<$ being interpreted as the usual ordering, are elementary equivalent, however, they are not isomorphic as no bijective map from $\mathbb{Q}$ to $\mathbb{R}$ exists.

Another example are algebraically closed fields. As we will see later, any two algebraically closed fields with the same characteristic are elementary equivalent in the language of rings $\mathcal{L}_r$, however, there are algebraically closed fields with the same characteristic but with different cardinalities.

Note however that any two finite elementary equivalent structures are isomorphic (see Proposition 2.1.1 of [13]).

1.1.5 Theories

As I said earlier, theories are a core concept in model theory. At the time, I informally defined a theory as a collection of ”properties” or ”axioms”, and intuitively, this is exactly what a theory is. The formal definition is the following:

Definition 1.1.29. Let $\mathcal{L}$ be a first-order language. A first-order $\mathcal{L}$-theory or simply an $\mathcal{L}$-theory T is simply a set of $\mathcal{L}$-sentences. Given an $\mathcal{L}$-structure $\mathcal{M}$, we say that $\mathcal{M}$ satisfies or models T, denoted by $\mathcal{M} \models T$ if $\mathcal{M} \models \phi$, for all $\phi \in T$.

We say that an $\mathcal{L}$-theory is satisfiable if there exists an $\mathcal{L}$-structure that models it.

Example 1.1.30.

- *In the language of groups $\mathcal{L}_g$ from example 1.1.7, the set*

$$\begin{aligned} T = \{&\forall x(x \cdot e = x \wedge e \cdot x = x), \\ &\forall x(x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e), \\ &\forall x \forall y \forall z[(x \cdot y) \cdot z = x \cdot (y \cdot z)]\} \end{aligned}$$

  is the first-order theory of groups. Given an $\mathcal{L}_g$-structure G, then $G \models T$ if and only if G is a group. If we were to add $\forall x \forall y(x \cdot y = y \cdot x)$ we would have the theory of abelian groups.

- *In the language of graphs $\mathcal{L} = \{R\}$, where R is a binary relation, the set*

$$\begin{aligned} T = \{&\forall x \neg R(x,x), \\ &\forall x \forall y(R(x,y) \to R(y,x))\} \end{aligned}$$

  is the first-order theory of simple graphs.

- *In the language of rings $\mathcal{L}_r$ from example 1.1.7, the set*

$$\begin{aligned} T = \{&\text{Axioms of additive commutative groups}, \\ &\forall x \forall y \forall z(x - y = z \leftrightarrow x = z + y), \\ &\forall x(x \cdot 1 = x \wedge 1 \cdot x = x), \\ &\forall x \forall y \forall z[(x \cdot y) \cdot z = x \cdot (y \cdot z)], \\ &\forall x \forall y \forall z(x \cdot (y + z) = x \cdot y + x \cdot z), \\ &\forall x \forall y \forall z((y + z) \cdot x = y \cdot x + z \cdot x)\} \end{aligned}$$

is the first-order theory of unitary rings, i.e. an $\mathcal{L}_r$-structure R models T if and only if R is a unitary ring. Note that the second axiom of T is necessary because we chose to include $-$ in the language, so we use this axiom to control how any interpretation of $-$ should behave.

- *In the language of rings $\mathcal{L}_r$, if we add to T the axioms $\forall x(x \neq 0 \rightarrow \exists y(x \cdot y = 1))$ and $\forall x \forall y(x \cdot y = y \cdot x)$, we get the first-order theory of fields.*

- *Now I will introduce an example that is very useful in many applications of model theory to ring/field theory. We are working with the language of rings $\mathcal{L}_r$, and for any $n \in \mathbb{N}$, consider the sentence ϕ_n given by*

  $$\forall a_0 \ldots \forall a_{n-1} \exists x \left(x^n + \sum_{i=1}^{n-1} a_i x^i = 0 \right)$$

  Then for any field F, we have that $F \models \phi_n$ if and only if every polynomial with coefficients in F and degree n has a root.

  We define $ACF =$ Theory of fields $\cup \{\phi_n : n \in \mathbb{N}\}$, so that $F \models ACF$ if and only if F is an algebraically closed field.

  Now, for any prime number p, let ψ_p be the sentence $\forall x(\underbrace{x + \ldots + x}_{p} = 0)$ and let $ACF_p = ACF \cup \{\psi_p\}$. Then $F \models ACF_p$ if and only if F is an algebraically closed field with characteristic p.

  Finally, let $ACF_0 = ACF \cup \{\neg\psi_p : p \in \mathbb{N}\}$. Then $F \models ACF_0$ if and only if F is an algebraically closed field with characteristic 0.

Given a structure $\mathcal{M}$, there is a very natural theory we can build, namely the theory of all sentences that are true in $\mathcal{M}$.

Definition 1.1.31. Given an $\mathcal{L}$-structure $\mathcal{M}$, we define the *full theory* of $\mathcal{M}$, denoted as $\text{Th}(\mathcal{M})$ as the theory consisting of all $\mathcal{L}$-sentences that are true in $\mathcal{M}$, i.e.

$$\text{Th}(\mathcal{M}) = \{\phi : \phi \text{ is an } \mathcal{L}\text{-sentence and } \mathcal{M} \models \phi\}$$

This theory has some very important properties, and we will mention it throughout the rest of this work. One interesting property, for example, is the following:

Proposition 1.1.32. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Then, for any $\mathcal{L}$-structure $\mathcal{N}$, we have that $\mathcal{M} \equiv \mathcal{N}$ if and only if Th$(\mathcal{M}) =$ Th$(\mathcal{N})$. Furthermore, $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathcal{N} \models$ Th$(\mathcal{M})$.*

Proof. The equivalence between $\mathcal{M} \equiv \mathcal{N}$ and $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$ is immediate from the definition.

For the second part, start by noting that if $\mathcal{M} \equiv \mathcal{N}$, then $\mathcal{N} \models \mathrm{Th}(\mathcal{N}) = \mathrm{Th}(\mathcal{M})$. Conversely, assume that $\mathcal{N} \models \mathrm{Th}(\mathcal{M})$. Then, for any sentence ϕ such that $\mathcal{M} \models \phi$, we have that $\mathcal{N} \models \phi$. Now assume that $\mathcal{N} \models \phi$ but $\mathcal{M} \not\models \phi$. Then, we would have that $\mathcal{M} \models \neg\phi$ however, this would imply that $\mathcal{N} \models \neg\phi$, which is a contradiction. Thus $\mathcal{M} \models \phi$. $\square$

Note that, in particular, this proposition implies that if $\mathcal{N}, \mathcal{K} \models \mathrm{Th}(\mathcal{M})$, then $\mathcal{N} \equiv \mathcal{K}$, as $\mathcal{N} \equiv \mathcal{M} \equiv \mathcal{K}$.

Theories with this property, i.e. such that $\mathcal{M}, \mathcal{N} \models T$ implies that $\mathcal{M} \equiv \mathcal{N}$ play a crucial role in model theory, and are known as complete theories.

Definition 1.1.33. We say that a satisfiable theory T is *complete* if any two models of T are elementary equivalent.

For example, it is possible to prove that the theory of dense linear orders without end-points, and the theory of algebraically closed fields of a given characteristic are complete, the latter due to Tarski.

We will discuss complete theories later once we have developed some more model-theoretic tools that allow us to prove some significant results about them.

Definition 1.1.34. Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence. We say that ϕ is a *logical consequence* of T, denoted by $T \models \phi$, if for any $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \models T$ implies that $\mathcal{M} \models \phi$. If Σ is a set of $\mathcal{L}$-sentences, we write $T \models \Sigma$ to denote that $T \models \phi$, for all $\phi \in \Sigma$.

Some examples of this include:

Example 1.1.35. *If T is the theory of groups from example 1.1.30, then*

$$T \models \forall x \forall y((x \cdot y = e \land y \cdot x = e) \to y = x^{-1})$$

or for example:

$$T \models \forall x(x \cdot x = x \to x = e)$$

I will end this chapter by introducing some common notations that we will use in the rest of this work.

Given a formula $\phi(w, \bar{v})$ and $n \in \mathbb{N}$, let $\exists^n x\phi(x, \bar{v})$ denote the formula:

$$\exists x_1 \ldots \exists x_n \left(\bigwedge_{i \neq j} \neg(x_i = x_j) \wedge \bigwedge_{i=1}^{n} \phi(x_i, \bar{v}) \wedge \forall y \left(\phi(y, \bar{v}) \rightarrow \bigvee_{i=1}^{n} (y = x_i) \right) \right)$$

This means that $\mathcal{M} \models \exists^n x\phi(x, \bar{v})[\bar{a}]$ if and only if there are exactly n elements $x_1, \ldots, x_n \in M$ such that $\mathcal{M} \models \phi(w, \bar{v})[x_i, \bar{a}]$.

For any $n \in \mathbb{N}$, let $\exists^{<n} x\phi(x, \bar{v})$ denote the formula

$$\bigvee_{k=1}^{n-1} \exists^k x\phi(x, \bar{v})$$

Let $\exists^{\leq n} x\phi(x, \bar{v})$ denote $\exists^{<n} x\phi(x, \bar{v}) \vee \exists^n x\phi(x, \bar{v})$. Similarly, let $\exists^{\geq n} x\phi(x, \bar{v})$ denote the formula:

$$\exists x_1 \ldots \exists x_n \bigwedge_{i=1}^{n} \phi(x_i, \bar{v})$$

and $\exists^{>n} x\phi(x, \bar{v})$ denote $\exists^{\geq n+1} x\phi(x, \bar{v})$.

If $\tau(v_1, \ldots, v_n)$ is a term, given terms $t_1, \ldots, t_n$, let $\tau(t_1, \ldots, t_n)$ be the term obtained from τ by substituting every occurrence of v_i by t_i, for all i. Note that

$$\tau(t_1, \ldots, t_n)^{\mathcal{M}}(\bar{a}) = \tau^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_n^{\mathcal{M}}(\bar{a}))$$

If $\phi(v_1, \ldots, v_n)$ is a formula and $t_1, \ldots, t_n$ are terms, let $\phi(t_1, \ldots, t_n)$ be the formula obtained from ϕ by substituting every free occurrence of v_i by t_i, for all i. In particular, if $t_i = c_i$, for some constant symbols c_i, then $\phi(c_1, \ldots, c_n)$ is the sentence obtained from substituting every free occurrence of v_i by c_i, for all i. Note that:

$$\mathcal{M} \models \phi(t_1, \ldots, t_n)[\bar{a}] \Leftrightarrow \mathcal{M} \models \phi[t_1^{\mathcal{M}}(\bar{a}), \ldots, t_n^{\mathcal{M}}(\bar{a})]$$

1.2. Definable and Algebraic Closure

I will now briefly introduce and discuss the concept of a definable set, which we will use extensively in the next chapter when dealing with o-minimality, so much so that even the definition of o-minimality requires the knowledge of what a definable set is. Intuitively, a definable set is a set that can be defined by an equation or a property in a first-order language $\mathcal{L}$.

Definition 1.2.1. Let $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure, $A \subseteq M$ and $X \subseteq M^n$. We say that X is A-*definable* , or simply definable if there exists an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_k)$

and $\bar{a} \in A^k$ such that

$$X = \{\bar{x} \in M^n : \mathcal{M} \models \phi[\bar{x}, \bar{a}]\}$$

It is common to denote the definable set X by $\phi(\mathcal{M}, \bar{a})$, or simply $X = \phi(\mathcal{M})$ if $A = \emptyset$.

If X is $\emptyset$-definable, we usually say that X is definable without parameters.

As a specific instance of 1.2.1, we say that an element $x \in M^n$ is definable if $\{x\}$ is definable. We say that a function $f : M^n \to M^k$ is definable if its graph $\Gamma(f) = \{(\bar{x}, f(\bar{x})) \in M^{n+k} : \bar{x} \in M^n\}$ is definable, and we say that an n-ary relation R in M is definable if it is definable as a subset R of M^n.

Example 1.2.2.

- *Consider the structure $R = (\mathbb{R}, <, 0)$. The set of non-negative real numbers is $\emptyset$-definable by the formula $\phi(x)$ given by $0 < x \vee x = 0$. However, if we consider the same structure, but this time without any constant symbols: $R = (\mathbb{R}, <)$, the set of non-negative real numbers would no longer be $\emptyset$-definable, but it would be 0-definable as we can use the parameter 0 to define it;*

- *Now, consider the structure $R = (\mathbb{R}, +, \cdot, 0, 1)$ and the formula $\phi(x, y)$ given by:*

$$\exists z (z \neq 0 \wedge y = x + z^2)$$

Then, for any $a, b \in \mathbb{R}$, we have that $a < b$ if and only if $R \models \phi[a, b]$, meaning that the order $<$ is $\emptyset$-definable.

The following result can be very useful to show that sets are not definable:

Proposition 1.2.3. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $X \subseteq M^n$ be an A-definable set. Then any $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes A pointwise, fixes X setwise.*

Proof. Let $\phi(\bar{v}, \bar{w})$ and $\bar{a} \in A^k$ such that $X = \{\bar{x} \in M^n : \mathcal{M} \models \phi[\bar{x}, \bar{a}]\}$ and let $f : M \to M$ be an isomorphism.

By what we saw in the proof of Proposition 1.1.28:

$$\mathcal{M} \models \phi[\bar{x}, \bar{a}] \Leftrightarrow \mathcal{M} \models \phi[f(\bar{x}), f(\bar{a})] \Leftrightarrow \mathcal{M} \models \phi[f(\bar{x}), \bar{a}]$$

so $\bar{x} \in X$ if and only if $f(\bar{x}) \in X$ which implies that $f(X) = X$. $\square$

As a particular case of this, if $x \in M$ is A-definable, then any automorphism of $\mathcal{M}$ that fixes A (and if A is empty, any isomorphism of $\mathcal{M}$) fixes x.

An example of the utility of Proposition 1.2.3, is the following:

Corollary 1.2.4. *In the language of rings, $\mathbb{R}$ is not definable in $\mathbb{C}$.*

The proof of this uses results from field theory that are outside of the scope of this work. The proof can be found in [5] as Corollary 1.3.6.

In field theory, an element of a field is said to be algebraic over a set A if it is the root of some polynomial with coefficients from A. Note, however, that an algebraic number is not necessarily the only root of such a polynomial. Now, for an element of that field to be definable over A, we would need to find a formula whose only "solution" would be that number. This is very strict when compared with the notion of algebraic number, whose defining formula may have a finite number of different solutions. This prompts the following definition:

Definition 1.2.5. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ and $x \in M$. We say that x is *algebraic over* A if there exists $\bar{a} \in A^k$ and a $\mathcal{L}$-formula $\phi(v, \bar{w})$ such that:

- $\mathcal{M} \models \phi[x, \bar{a}]$;
- The set $\{y \in M : \mathcal{M} \models \phi[y, \bar{a}]\}$ is finite.

This is equivalent to saying that there exists a formula $\phi(v, \bar{w})$ and $\bar{a} \in A$ such that $\mathcal{M} \models (\phi(v, \bar{w}) \wedge \exists^n z \phi(z, \bar{w}))[x, \bar{a}]$, for some $n \in \mathbb{N}$.

The notion of definable and algebraic sets gives rise to the following two closure operators:

Definition 1.2.6. Let $\mathcal{M}$ be a structure and $A \subseteq M$. The *definable closure* of A, denoted by dcl(A) is the set of all A-definable elements of M. The *algebraic closure* of A, denoted by acl(A) is the set of all elements of M algebraic over A

These two operators have the following basic properties:

Proposition 1.2.7. *Let $\mathcal{M}$ be a structure and $A \subseteq M$. Then:*

1. *dcl$(A) \subseteq$ acl(A);*
2. *If $B \subseteq A$, then dcl$(B) \subseteq$ dcl(A) and acl$(B) \subseteq$ acl(A);*

3. *The operators dcl and acl are idempotent;*

Proof. 1) If $x \in \text{dcl}(A)$ then there exists a formula $\phi(v, \varpi)$ and $\bar{a} \in A^k$ such that $\{x\} = \{y \in M : \mathcal{M} \models \phi[y, \bar{a}]\}$, so that $x \in \text{acl}(A)$.

2) Let $\{x\}$ is B-definable and $B \subseteq A$. Any parameter from B is a parameter from A, so $\{x\}$ is A-definable. The proof that $\text{acl}(B) \subseteq \text{acl}(A)$ is analogous.

3) It is clear that $A \subseteq \text{dcl}(A)$, so by (2), $\text{dcl}(A) \subseteq \text{dcl}(\text{dcl}(A))$. Now, let $x \in \text{dcl}(\text{dcl}(A))$. This means that there exists a formula $\phi(v, \varpi)$ and $a_1, \ldots, a_n \in \text{dcl}(A)$ such that $\mathcal{M} \models \phi[y, a_1, \ldots, a_n]$ if and only if $y = x$. Now, because for each $i \in \{1, \ldots, n\}$ we have $a_i \in \text{dcl}(A)$, there exists a formula $\psi_i(v, \varpi)$ and $b_i^1, \ldots, b_i^k \in A$ such that $\mathcal{M} \models \psi_i[y, b_i^1, \ldots, b_i^k]$ if and only if $y = a_i$. We will assume that for all i we use k parameters from A to define a_i for simplicity, but in reality, each a_i could take a different number of parameters from A, however, this does not invalidate the proof. Consider now the formula $\theta(v, \varpi_1, \ldots, \varpi_n)$ given by:

$$\exists v_1 \ldots \exists v_n \left(\bigwedge_{i=1}^{n} \psi_i(v_i, \varpi_i) \wedge \phi(v, v_1, \ldots, v_n) \right)$$

Then, $\{x\} = \{y \in M : \mathcal{M} \models \theta(y, b_1^1, \ldots, b_1^k, \ldots, b_n^1, \ldots, b_n^k)\}$, proving that $x \in \text{dcl}(A)$.

Similarly, we have $A \subseteq \text{acl}(A)$ and thus $\text{acl}(A) \subseteq \text{acl}(\text{acl}(A))$.

Let $x \in \text{acl}(\text{acl}(A))$. There exists $\phi(v, \varpi)$, $n \in \mathbb{N}$ and $a_1, \ldots, a_k \in \text{acl}(A)$ such that: $\mathcal{M} \models (\phi(v, \varpi) \wedge \exists^n z \phi(z, \varpi))[x, \bar{a}]$. For each $a_i \in \text{acl}(A)$, there exists $\psi_i(v, \varpi)$, $n_i \in \mathbb{N}$ and $\bar{b}_i = (b_i^1, \ldots, b_i^j) \in A^j$ such that $\mathcal{M} \models (\psi_i(v, \varpi) \wedge \exists^{n_i} z \psi_i(z, \varpi))[a_i, \bar{b}_i]$. Now, consider the formula $\theta(v, \varpi_1, \ldots, \varpi_n)$ given by:

$$\exists v_1 \ldots \exists v_k \left(\bigwedge_{i=1}^{k} \psi_i(v_i, \varpi_i) \wedge \phi(v, v_1, \ldots, v_k) \wedge \exists^n z \phi(z, v_1, \ldots, v_k) \right)$$

Then $\mathcal{M} \models (\theta(v, \varpi_1, \ldots, \varpi_n) \wedge \exists^p z \theta(z, \varpi_1, \ldots, \varpi_n))[x, \bar{b}_1, \ldots, \bar{b}_k]$, for some $p \in \mathbb{N}$, because it has at most $n \cdot n_1 \cdot \ldots \cdot n_k$ solutions, meaning that $x \in \text{acl}(A)$ $\square$

1.3. The Compactness Theorem

The compactness theorem is the cornerstone of model theory, so much so that we will regularly use it explicitly or implicitly throughout the rest of this work. There are a few known ways of proving the compactness theorem, each with its own set of advantages and disadvantages, and each different approach reveals a different perspective about the deep nature of this theorem and how it relates to other concepts in logic and mathematics. The one I choose to follow is due to Leon Henkin and it is known as Henkin's Construction.

1.3.1 Henkin Construction

The goal of this subsection is to prove the compactness theorem, which states the following:

Theorem 1.3.1 (The compactness theorem)**.** *A theory T is satisfiable if and only if every finite subset of T is satisfiable.*

Proving one of these implications is trivial, namely: if T is an $\mathcal{L}$-theory, and there exists an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models T$, then for any finite subset $\Delta \subseteq T$, we have $\mathcal{M} \models \Delta$.

The purpose of the rest of this section is to prove that, if for every finite subset $\Delta \subseteq T$ there exists $\mathcal{M}_\Delta$ such that $\mathcal{M}_\Delta \models \Delta$, then there exists a structure that satisfies all of T. We say that T is finitely satisfiable if every finite subset of T is satisfiable. Our goal is then to start with a finitely satisfiable theory T and construct a model for T.

We begin with the following definitions:

Definition 1.3.2. We say that a theory T is *maximal* if for all sentences ϕ, either $\phi \in T$ or $\neg\phi \in T$.

Definition 1.3.3. We say that a theory T has the *witness property* if for every formula $\phi(v)$ there exists a constant symbol c such that $(\exists v\phi(v)) \to \phi(c) \in T$.

Our plan for proving the compactness theorem will roughly be the following:

1. Start with a finitely satisfiable theory T;
2. Prove that any finitely satisfiable, maximal theory with the witness property is satisfiable;
3. Extend T to a finitely satisfiable, maximal theory with the witness property;
4. From this, conclude that T is satisfiable.

We start by proving that following technical lemma.

Lemma 1.3.4. *Let T be a finitely satisfiable $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence. Then either $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable.*

Proof. Assume that $T \cup \{\phi\}$ is not finitely satisfiable. Then there exists a finite subset $\Gamma \subseteq T$ such that $\Gamma \cup \{\phi\}$ has no models. Let $\Delta \subseteq T \cup \{\neg\phi\}$ be any finite subset. If $\neg\phi \notin \Delta$, then Δ has a model as T is finitely satisfiable. If $\neg\phi \in \Delta$, write $\Delta = \Delta' \cup \{\neg\phi\}$, for some finite $\Delta' \subseteq T$. $\Delta' \cup \Gamma$ is a finite subset of T and as such, there exists $\mathcal{M}$ that models $\Delta' \cup \Gamma$. If $\mathcal{M} \models \phi$, then it would be a model of $\Gamma \cup \{\phi\}$, so $\mathcal{M} \models \neg\phi$, and in particular, $\mathcal{M} \models \Delta' \cup \{\neg\phi\} = \Delta$.

Using the same argument and the fact that $\neg\neg\phi$ is equivalent to ϕ, we get that if $T \cup \{\neg\phi\}$ is not finitely satisfiable, then $T \cup \{\phi\}$ is. □

This lemma together with Zorn's lemma allows us to prove the following:

Corollary 1.3.5. *If T is a finitely satisfiable $\mathcal{L}$-theory, then there is a maximal finitely satisfiable $\mathcal{L}$-theory T' such that $T \subseteq T'$.*

Proof. Consider the set

$$\Gamma = \{T^* : T \subseteq T^* \text{ and } T^* \text{ is finitely satisfiable}\}$$

Let C be a chain in Γ with respect to inclusion, let $T_0 = \bigcup_{K \in C} K$ and let $\Delta \subseteq T_0$ be finite. Then $\Delta \subseteq K$, for some $K \in C$, and because K is finitely satisfiable, Δ has a model, meaning that T_0 is finitely satisfiable. By Zorn's lemma, Γ has a maximal element T^* with respect to inclusion. As $T^* \in \Gamma$, we have that T^* is finitely satisfiable, and $T \subseteq T^*$, so the only thing left to verify is that T^* is maximal. Let ϕ be any $\mathcal{L}$-sentence. Then, by Lemma 1.3.4, either $T^* \cup \{\phi\}$ or $T^* \cup \{\neg\phi\}$ is finitely satisfiable. In the first case, as $T^* \subseteq T^* \cup \{\phi\}$, we have that $\phi \in T^*$, by maximally. If this is not the case, by the same argument, we have $\neg\phi \in T^*$. □

This next simple lemma will also be very useful in constructing a model for a finitely satisfiable theory T.

Lemma 1.3.6. *Let T be a maximal and finitely satisfiable theory. Let ϕ be a sentence and $\Delta \subseteq T$ be a finite subset. Then, if $\Delta \models \phi$, we have that $\phi \in T$.*

Proof. Because T is maximal, if $\phi \notin T$, then $\neg\phi \in T$. So $\Delta \cup \{\neg\phi\}$ is a finite subset of T and because T is finitely satisfiable, there exists a structure $\mathcal{M}$ such that $\mathcal{M} \models \Delta \cup \{\neg\phi\}$, which is a contradiction because $\Delta \models \phi$. □

In particular, we have the following corollary.

Corollary 1.3.7. *Let T be a maximal, finitely satisfiable theory with the witness property, and let $\phi(v)$ be a formula. If $\exists x\phi(x) \in T$, then there exists a constant symbol c such that $\phi(c) \in T$.*

Proof. Because T has the witness property, there exists a constant symbol c such that $((\exists x\phi(x)) \to \phi(c)) \in T$. If $\exists x\phi(x) \in T$, because

$$\{\exists x\phi(x), ((\exists x\phi(x)) \to \phi(c))\} \models \phi(c)$$

by Lemma 1.3.6, we have that $\phi(c) \in T$. □

The following lemma is the crux of Henkin's Construction. Although the proof is long and has lots of little details, after proving this lemma, Henkin's proof of the compactness theorem is almost finished.

Lemma 1.3.8 (Henkin's Lemma)**.** *Let T be a maximal, finitely satisfiable $\mathcal{L}$-theory with the witness property. Then T has a model.*

In particular, if $\mathcal{L}$ has at most κ constant symbols, for some infinite cardinal κ, then there exists $\mathcal{M} \models T$ such $|M| \leq \kappa$.

Proof. Let C be the set of constant symbols in $\mathcal{L}$, and consider the relation $\sim$ in C given by: $a \sim b$ if and only if $(a = b) \in T$.

We start by verifying that $\sim$ is an equivalence relation:

1) Let $a \in C$. As $\emptyset \models a = a$, by Lemma 1.3.6, we have $a = a \in T$;

2) Let $a, b \in C$ such that $a = b \in T$. As $\{a = b\} \models b = a$, by Lemma 1.3.6 we have $(b = a) \in T$;

3) Let $a, b, c \in T$ such that $a = b, b = c \in T$. As $\{a = b, b = c\} \models a = c$, by Lemma 1.3.6, we have $(a = c) \in T$.

Let $M = C/\sim$. We will build a model $\mathcal{M}$ of T with universe M, and note that, if $|C| \leq \kappa$, then $|M| \leq \kappa$. If c is a constant symbol, we let $c^{\mathcal{M}}$ be the equivalence class of c, which I will denote by c^*. Now let R be an n-ary relational symbol. We interpret R as follows:

$$(c_1^*, \dots, c_n^*) \in R^{\mathcal{M}} \text{ if and only if } R(c_1, \dots, c_n) \in T$$

Before we continue, we need to make sure that $R^{\mathcal{M}}$ is well-defined, i.e. if $R(c_1,\ldots,c_n) \in T$ and $c_i \sim d_i$, then $R(d_1,\ldots,d_n) \in T$. This is true because

$$\{R(c_1,\ldots,c_n), c_1 = d_1,\ldots,c_n = d_n\} \models R(d_1,\ldots,d_n)$$

and thus by Lemma 1.3.6, we have $R(d_1,\ldots,d_n) \in T$.

Let f be an n-ary functional symbol. We define the interpretation of f in $\mathcal{M}$ as:

$$f^{\mathcal{M}}(c_1^*,\ldots,c_n^*) = d^* \text{ if and only if } f(c_1,\ldots,c_n) = d \in T$$

There are a few things we need to verify to ensure that $f^{\mathcal{M}}$ is well-defined.

1) Firstly, we need to make sure that for all $c_1,\ldots,c_n \in C$, there exists a constant symbol d such that $f(c_1,\ldots,c_n) = d \in T$. Consider the formula $\phi(v)$ given by $f(c_1,\ldots,c_n) = v$. As $\emptyset \models \exists v\phi(v)$, by Lemma 1.3.6, we have that $\exists v\phi(v) \in T$, and by Corollary 1.3.7, there exists $d \in C$ such that $\phi(d) \in T$;

2) Now let $c_1,\ldots,c_n,d_1,\ldots,d_n \in C$ such that $c_i \sim d_i$. We wish to show that if $f(c_1,\ldots,c_n) = e \in T$ and $f(d_1,\ldots,d_n) = k \in T$, then $e^* = k^*$, i.e. $e \sim k$. We have that

$$\{c_1 = d_1,\ldots,c_n = d_n\} \models f(c_1,\ldots,c_n) = f(d_1,\ldots,d_n)$$

so by Lemma 1.3.6, it follows that $f(c_1,\ldots,c_n) = f(d_1,\ldots,d_n) \in T$. Finally, because

$$\{f(c_1,\ldots,c_n) = f(d_1,\ldots,d_n), f(c_1,\ldots,c_n) = e, f(d_1,\ldots,d_n) = k\} \models e = k$$

by Lemma 1.3.6, we conclude that $e = k \in T$, i.e. $e \sim k$.

The last step of the proof is verifying that $\mathcal{M}$ is a model of T. To facilitate the process, we begin by showing that terms in $\mathcal{M}$ are well-behaved, that is: given a term $\tau(v_1,\ldots,v_n)$ and $c_1,\ldots,c_n,d \in C$,

$$\tau^{\mathcal{M}}(c_1^*,\ldots,c_n^*) = d^* \text{ if and only if } \tau(c_1,\ldots,c_n) = d \in T$$

We will use induction on terms:

1) Let $\tau(v_1,\ldots,v_n) = k$ for some constant symbol k. If $\tau^{\mathcal{M}}(c_1^*,\ldots,c_n^*) = d^*$, then $k^* = d^*$, meaning that $k \sim d$ and thus $k = d \in T$.

On the other hand, if $\tau(c_1,\ldots,c_n) = d \in T$, then $k \sim d$ and thus $\tau^{\mathcal{M}}(c_1^*,\ldots,c_n^*) = d^*$.

2) Let $\tau(v_1, \ldots, v_n) = v_p$. If $\tau^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$, i.e. $c_p^* = d^*$, then $c_p = d \in T$, meaning that $\tau(c_1, \ldots, c_n) = d \in T$.

On the other hand if $\tau(c_1, \ldots, c_n) = d \in T$, i.e. $c_p = d \in T$, then $c_i^* = d^*$, meaning that $\tau^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$.

3) Let $\tau = f(t_1(v_1, \ldots, v_n), \ldots, t_k(v_1, \ldots, v_n))$, for some functional symbol f and terms $t_1, \ldots, t_k$. Assume that $\tau^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$, i.e.:

$$f^{\mathcal{M}}(t_1^{\mathcal{M}}(c_1^*, \ldots, c_n^*), \ldots, t_k^{\mathcal{M}}(c_1^*, \ldots, c_n^*)) = d^*$$

For each $i = 1, \ldots, k$, let $e_i \in C$ be a representative of the equivalence class $t_i^{\mathcal{M}}(c_1^*, \ldots, c_n^*)$. Then, we have

$$f^{\mathcal{M}}(e_1^*, \ldots, e_k^*) = d^*$$

By the definition of the interpretation of functional symbols in $\mathcal{M}$, we can conclude that $f(e_1, \ldots, e_k) = d \in T$. We also have that $t_i^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = e_i^*$, so by the induction hypothesis, $t_i(c_1, \ldots, c_n) = e_i \in T$, for all $i = 1, \ldots, k$. Additionally, because

$$\{f(e_1, ..., e_k) = d\} \cup \{t_i(c_1, ..., c_n) = e_i : i = 1, ..., k\} \models f(t_1(c_1, ..., c_n), ..., t_k(c_1, ..., c_n)) = d$$

by Lemma 1.3.6 we conclude that $f(t_1(c_1, \ldots, c_n), \ldots, t_k(c_1, \ldots, c_n)) = d \in T$.

On the other hand, assume now

$$f(t_1(c_1, \ldots, c_n), \ldots, t_k(c_1, \ldots, c_n)) = d \in T$$

For each $i = 1, \ldots, k$, consider the formula $\phi_i(v)$ given by $t_i(c_1, \ldots, c_n) = v$. As $\emptyset \models \exists x \phi_i(x)$, by Lemma 1.3.6, we have $\exists x \phi_i(x) \in T$, and by Corollary 1.3.7, for all $i = 1, \ldots, k$, there exists $e_i \in C$ such that $\phi(e_i) \in T$, i.e. $t_i(c_1, \ldots, c_n) = e_i \in T$. By the induction hypothesis, $t_i^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = e_i^*$. Because

$$\{f(t_1(c_1, ..., c_n), ..., t_k(c_1, ..., c_n)) = d\} \cup \{t_i(c_1, ..., c_n) = e_i, i = 1, ..., k\} \models f(e_1, ..., e_k) = d$$

by Lemma 1.3.6, we have that $f(e_1, \ldots, e_k) = d$. By the definition of $f^{\mathcal{M}}$, we have that $f^{\mathcal{M}}(e_1^*, \ldots, e_k^*) = d^*$ and thus $f^{\mathcal{M}}(t_1^{\mathcal{M}}(c_1^*, \ldots, c_n^*), \ldots, t_k^{\mathcal{M}}(c_1^*, \ldots, c_n^*)) = d^*$.

Now We will prove that given a formula $\phi(v_1, \ldots, v_n)$ and $c_1, \ldots, c_n \in C$, then $\mathcal{M} \models \phi[c_1^*, \ldots, c_n^*]$ if and only if $\phi(c_1, \ldots, c_n) \in T$. We use induction on formulas:

1) If ϕ is $t_1 = t_2$. By Lemma 1.3.6 and Corollary 1.3.7, we can find $d_1, d_2 \in C$ such that $t_1(\bar{c}) = d_1, t_2(\bar{c}) = d_2 \in T$, and thus $t_1^{\mathcal{M}}(\bar{c}^*) = d_1^*$ and $t_2^{\mathcal{M}}(\bar{c}^*) = d_2^*$. Then

$$\begin{aligned}
\mathcal{M} \models (t_1 = t_2)[\bar{c}^*] &\Leftrightarrow t_1^{\mathcal{M}}(\bar{c}^*) = t_2^{\mathcal{M}}(\bar{c}^*) \\
&\Leftrightarrow d_1^* = d_2^* \\
&\Leftrightarrow d_1 = d_2 \in T \\
&\Leftrightarrow t_1(\bar{c}) = t_2(\bar{c}) \in T \qquad \text{(Lemma 1.3.6)}
\end{aligned}$$

2) If ϕ is $R(t_1, \ldots, t_k)$. By the same argument we did in the previous case, there exists $d_1, \ldots, d_k \in C$ such that $t_i(\bar{c}) = d_i \in T$ and $t_i^{\mathcal{M}}(\bar{c}^*) = d_i^*$. So

$$\begin{aligned}
\mathcal{M} \models \phi[\bar{c}^*] &\Leftrightarrow (t_1^{\mathcal{M}}(\bar{c}^*), \ldots, t_k^{\mathcal{M}}(\bar{c}^*)) \in R^{\mathcal{M}} \\
&\Leftrightarrow (d_1^*, \ldots, d_k^*) \in R^{\mathcal{M}} \\
&\Leftrightarrow R(d_1, \ldots, d_k) \in T \qquad \text{(Definition of } R^{\mathcal{M}}\text{)} \\
&\Leftrightarrow R(t_1(\bar{c}), \ldots, t_k(\bar{c})) \in T \qquad \text{(Lemma 1.3.6)}
\end{aligned}$$

3) If ϕ is $\neg\psi(\bar{v})$, for some ψ, then:

$$\begin{aligned}
\mathcal{M} \models \phi[\bar{c}^*] &\Leftrightarrow \mathcal{M} \not\models \psi[\bar{c}^*] \\
&\Leftrightarrow \psi(\bar{c}) \notin T \qquad \text{(Induction Hypothesis)} \\
&\Leftrightarrow \neg\psi(\bar{c}) \in T \qquad \text{(T is complete)} \\
&\Leftrightarrow \phi(\bar{c}) \in T
\end{aligned}$$

4) If ϕ is $\psi \wedge \chi$, then the proof can be done similarly to (3) just by using the induction hypothesis.

5) If ϕ is $\exists x\psi(x, v_1, \ldots, v_n)$. Then

$$\begin{aligned}
\mathcal{M} \models \phi[\bar{c}^*] &\Leftrightarrow \mathcal{M} \models \psi[d^*, \bar{c}^*] \qquad \text{(for some } d \in C\text{)} \\
&\Leftrightarrow \psi(d, \bar{c}) \in T \qquad \text{(for some } d \in C\text{)} \\
&\Leftrightarrow \exists x\psi(x, \bar{c}) \in T \qquad (\{\psi(d, \bar{c})\} \models \exists x\psi(x, \bar{c}) \text{ and Lemma 1.3.6)}
\end{aligned}$$

This concludes the induction. In particular, for any sentence ϕ, we have shown that $\mathcal{M} \models \phi$ if and only if $\phi \in T$, so $\mathcal{M} \models T$. $\square$

Henkin's Lemma asserts that any maximal finitely satisfiable theory with the witness property is satisfiable. With this result, the only thing left to prove is that we can extend any finitely satisfiable theory to a maximal one with the witness property.

Lemma 1.3.9 (Lindenbaum's Lemma)**.** *Let T be a finitely satisfiable $\mathcal{L}$-theory. Then there is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and an $\mathcal{L}^*$-theory $T^* \supseteq T$ such that T^* is finitely satisfiable, and any $\mathcal{L}^*$-theory extending T^* has the witness property. Moreover, we can choose $\mathcal{L}^*$ such that $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.*

Proof. The idea of the proof is to gradually expand the language $\mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \dots$ and the theory $T = T_0 \subseteq T_1 \subseteq \dots$ so that for all $i \in \mathbb{N}$, T_i is an $\mathcal{L}_i$-theory, and when building T_{i+1} from T_i, we add all potential witnesses we could be missing.

We begin with $\mathcal{L}_0 = \mathcal{L}$ and $T_0 = T$. Now, let us assume that we already built $\mathcal{L}_i$ and T_i. For each $\mathcal{L}_i$-formula $\phi(x)$, let c_ϕ be a new constant symbol and let

$$\mathcal{L}_{i+1} = \mathcal{L}_i \cup \{c_\phi : \phi(x) \text{ is an } \mathcal{L}_i\text{-formula}\}$$

Now, let

$$T_{i+1} = T_i \cup \{(\exists x \phi(x)) \rightarrow \phi(c_\phi) : \phi(x) \text{ is an } \mathcal{L}_i\text{-formula}\}$$

We start by showing that each T_i is finitely satisfiable using induction. $T_0 = T$ is finitely satisfiable by definition. Let T_i be finitely satisfiable and let $\Delta \subseteq T_{i+1}$ be a finite subset. To simplify the notation, let χ_ϕ denote the sentence $(\exists x \phi(x)) \rightarrow \phi(c_\phi)$. So $\Delta = \Delta' \cup \{\chi_{\phi_1}, \dots, \chi_{\phi_n}\}$, for some $\Delta' \subseteq T_i$ and $\phi_1, \dots, \phi_n$ $\mathcal{L}_i$-formulas. As T_i is finitely satisfiable, there exists an $\mathcal{L}_i$-structure $\mathcal{M}$ such that $\mathcal{M} \models \Delta'$. The idea now is to find a way to correctly interpret the constant symbols $c_{\phi_1}, \dots, c_{\phi_k}$ in order to turn $\mathcal{M}$ into an $\mathcal{L}_{i+1}$-structure that models Δ (note that the interpretation of the other constant symbols that do not appear in Δ do not matter as they do not change whether $\mathcal{M}$ models Δ or not). For any $k = 1, \dots, n$, if $\mathcal{M} \models \exists x \phi_k(x)$, let $m_k \in M$ be an element such that $\mathcal{M} \models \phi_k[m_k]$, and define $c_{\phi_k}^{\mathcal{M}} = m_k$. If $\mathcal{M} \not\models \exists x \phi_k(x)$, let $c_{\phi_k}^{\mathcal{M}}$ be any arbitrary element of M. As to all the other constant symbols in $\mathcal{L}_{i+1}$ other than $c_{\phi_1}, \dots, c_{\phi_k}$, we may choose them arbitrarily. So now, we have that $\mathcal{M}$ is an $\mathcal{L}_{i+1}$-structure such that for all $k = 1, \dots, n$, $\mathcal{M} \models \chi_{\phi_k}$, and thus $\mathcal{M} \models \Delta$.

Let $\mathcal{L}^* = \bigcup_{n \in \omega} \mathcal{L}_n$ and $T^* = \bigcup_{n \in \omega} T_n$. For any finite subset $\Delta \subseteq T^*$, there exists an $i \in \mathbb{N}$ such that $\Delta \subseteq T_i$, and because T_i is finitely satisfiable, Δ has a model. Note also that by construction, T^* has the witness property.

Let $\Sigma \supseteq T^*$ be any $\mathcal{L}^*$-theory extending T^*. Let $\phi(v)$ be an $\mathcal{L}^*$-formula. Because $\phi(v)$ is finite, it only uses finitely many constant symbols from $\mathcal{L}^*$, meaning that it exists an $k \in \mathbb{N}$ such that $\phi(v)$ is actually an $\mathcal{L}_k$-formula. This means that $(\exists x \phi(x)) \to \phi(c_\phi)$ is an element of $T_{k+1} \subseteq T^* \subseteq \Sigma$. Thus Σ has the witness property.

As for the cardinality of $\mathcal{L}^*$, we first claim that for each $i \in \mathbb{N}_{>0}$, $|\mathcal{L}_i| = |\mathcal{L}| + \aleph_0$. For each i, the cardinality of the set of $\mathcal{L}_i$-sentences is $|\mathcal{L}_i| + \aleph_0$, thus $|\mathcal{L}_{i+1}| = |\mathcal{L}_i| + |\mathcal{L}_i| + \aleph_0 = |\mathcal{L}_i| + \aleph_0$. As $|\mathcal{L}_1| = |\mathcal{L}| + \aleph_0$, we conclude that $|\mathcal{L}_i| = |\mathcal{L}| + \aleph_0$ for all i. As $\mathcal{L}^* = \bigcup_{n \in \omega} \mathcal{L}_n$, we have that $|\mathcal{L}^*| = \aleph_0(|\mathcal{L}| + \aleph_0) = |\mathcal{L}| + \aleph_0$. □

With all of this, we can finally prove the compactness theorem, and in fact, by Henkin's Lemma, we can additionally, to some degree control the cardinality of the model.

Theorem 1.3.10. *Let T be a finitely satisfiable $\mathcal{L}$-theory, and κ be an infinite cardinal such that $|\mathcal{L}| \leq \kappa$. Then T has a model with cardinality at most κ.*

Proof. By Lemma 1.3.9, there exists a language $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ such that T^* is finitely satisfiable and any expansion of T^* has the witness property. By Corollary 1.3.5, there exists a maximal finitely satisfiable $\mathcal{L}^*$-theory T' such that $T^* \subseteq T'$, and as such, T' has the witness property. Because T' is maximal, finitely satisfiable and has the witness property, we can apply Lemma 1.3.8 to build an $\mathcal{L}^*$-structure $\mathcal{M}$ that models T', and in particular, as $T \subseteq T'$, we conclude that $\mathcal{M} \models T$. Ignoring all additional constants from $\mathcal{L}^*$ that are not already in $\mathcal{L}$, we get an $\mathcal{L}$-structure that models T. Note also that the construction we did in Lemma 1.3.8, guarantees that $|\mathcal{M}| \leq \kappa$. □

In particular, if $\mathcal{L}$ is countable and T is a finitely satisfiable $\mathcal{L}$-theory, then T has a countable model.

1.3.2 Examples

Before continuing to the next section, We will explore some basic examples of direct applications of the compactness theorem.

Corollary 1.3.11. *Let T be a theory that has arbitrarily large finite models, then T has an infinite model.*

Proof. For each $n \in \mathbb{N}$, let ϕ_n be the sentence

$$\exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j$$

Then $\mathcal{M} \models \phi_n$ if and only if $\mathcal{M}$ has at least n elements. Now, consider the theory $T \cup \{\phi_n : n \in \mathbb{N}\}$. Let $\Delta \subseteq T \cup \{\phi_n : n \in \mathbb{N}\}$ be finite, and let $k \in \mathbb{N}$ be the greatest natural number such that $\phi_k \in \Delta$. As T has arbitrarily large finite models, there exists $\mathcal{M}' \models T$ such that $\mathcal{M}'$ has more than k elements, meaning that $\mathcal{M}' \models \Delta$. By the compactness theorem, there exists $\mathcal{M}$ that models $T \cup \{\phi_n : n \in \mathbb{N}\}$, meaning that $\mathcal{M} \models T$ and $\mathcal{M}$ is infinite. □

Corollary 1.3.12. *Consider the language $\mathcal{L} = \{\cdot, +, <, 0, 1\}$. Then there exists an $\mathcal{L}$-structure elementarily equivalent to the natural numbers that has an element greater than every natural number.*

Proof. Consider the new language $\mathcal{L}^* = \mathcal{L} \cup \{c\}$. For each $n \in \mathbb{N}$, let ϕ_n be the sentence $\underbrace{1 + 1 + \ldots + 1}_{n} < c$, and in particular, $\mathcal{M} \models \phi_k$ if and only if $c^{\mathcal{M}} > k$. Consider now the theory

$$\mathrm{Th}(\mathbb{N}) \cup \{\phi_n : n \in \mathbb{N}\}$$

Let $\Delta \subseteq \mathrm{Th}(\mathbb{N}) \cup \{\phi_n : n \in \mathbb{N}\}$ be finite, and let k be the greatest natural number such that $\phi_k \in \Delta$. Then, by interpreting the new constant symbol c in $\mathbb{N}$ as the number $k+1$, we have that $\mathbb{N} \models \Delta$. By compactness, there exists $\mathcal{M}$ that models $\mathrm{Th}(\mathbb{N}) \cup \{\phi_n : n \in \mathbb{N}\}$, and as such, $\mathcal{M}$ is elementarily equivalent to $\mathbb{N}$ and the element $c^{\mathcal{M}} \in M$ is greater than any natural number, as $\mathcal{M} \models \phi_n$ for all $n \in \mathbb{N}$. □

Finally, we have the following useful lemma:

Lemma 1.3.13. *Let T be a theory and ϕ be a sentence. Then $T \models \phi$ if and only if there exists a finite subset $\Delta \subseteq T$ such that $\Delta \models \phi$.*

Proof. If $\Delta \models \phi$, for some finite subset Δ of T, then trivially we have that $T \models \phi$.

On the other hand, assume that $T \models \phi$ and for all finite $\Delta \subseteq T$ we have $\Delta \not\models \phi$. This implies that $T \cup \{\neg\phi\}$ is satisfiable, because given any finite subset $\Gamma = \Delta \cup \{\neg\phi\}$ of $T \cup \{\neg\phi\}$ with $\Delta \subseteq T$, by our hypothesis, we have that $\Delta \not\models \phi$ and therefore, there exists $\mathcal{M} \models \Delta$ such that $\mathcal{M} \not\models \phi$. By the compactness theorem, there exists a model of $T \cup \{\neg\phi\}$, which contradicts the fact that $T \models \phi$. □

This has very interesting implications, for example, it allows us to give a short proof that in the language $\mathcal{L} = \emptyset$, there is no first-order sentence ϕ such that $\mathcal{M} \models \phi$ if and only if $\mathcal{M}$ is infinite. Assume that such formula ϕ exists. For each natural number n, let ϕ_n be the formula we used in the proof of Corollary 1.3.11. Then $\{\phi_n : n \in \mathbb{N}\} \models \phi$. But that would imply that there is a finite subset $\Delta \subseteq \{\phi_n : n \in \mathbb{N}\}$ such that $\Delta \models \phi$, which is obviously false.

1.4. Löwenheim–Skolem Theorem

Another central result in model theory is the Löwenheim–Skolem Theorem, which is often divided into two theorems: the Downward Löwenheim–Skolem Theorem and the Upward Löwenheim–Skolem Theorem. As we will see, the proof is nothing more that a clever application of the compactness theorem. Nonetheless, the consequences and applications of the Löwenheim–Skolem Theorem are important enough for it to be considered its own theorem instead of just a corollary of the compactness theorem.

We begin by defining an important class of maps:

Definition 1.4.1. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. We say that a map $f : M \to N$ is elementary, if for any $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n)$ and for any $\bar{a} \in M^n$, we have

$$\mathcal{M} \models \phi[\bar{a}] \Leftrightarrow \mathcal{N} \models \phi[f(\bar{a})]$$

Note that, by the proof of Proposition 1.1.28, any isomorphism is an elementary map.

Note also that, if there exists an elementary map from $\mathcal{M}$ to $\mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.

Proposition 1.4.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures and let $f : M \to N$ be an elementary map. Then f is an embedding.*

Proof. We start by showing that f is injective. Let $x, y \in M$ and consider the formula $\phi(a, b) = \neg(a = b)$. If $x \neq y$, then $\mathcal{M} \models \phi[x, y]$, and f being an elementary map implies that $\mathcal{N} \models \phi[f(x), f(y)]$, meaning that $f(x) \neq f(y)$.

Now let c be any constant symbol and consider the formula $\phi(v_1) = (v_1 = c)$. Then we have that $\mathcal{M} \models \phi[c^{\mathcal{M}}]$ and because f is an elementary map, we have that $\mathcal{N} \models \phi[f(c^{\mathcal{M}})]$ meaning that $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

Now let g be an n-ary functional symbol and consider the formula $\phi(v_1,\dots,v_n,b)$ given by $g(v_1,\dots,v_n)=b$. Let $x_1,\dots,x_n\in M$. Then

$$\mathcal{M}\models\phi[x_1,\dots,x_n,g^{\mathcal{M}}(x_1,\dots,x_n)]$$

Again, because f is an elementary map, we have that

$$\mathcal{N}\models\phi[f(x_1),\dots,f(x_n),f(g^{\mathcal{M}}(x_1,\dots,x_n))]$$

meaning that

$$g^{\mathcal{N}}(f(x_1),\dots,f(x_n))=f(g^{\mathcal{M}}(x_1,\dots,x_n))$$

for any $x_1,\dots,x_n\in M$.

To conclude that proof, let R be an n-ary relational symbol, and consider the formula $\phi(v_1,\dots,v_n)$ given by $R(v_1,\dots,v_n)$. For any $x_1,\dots,x_n$, we have that:

$$\begin{aligned}(x_1,\dots,x_n)\in R^{\mathcal{M}} &\Leftrightarrow \mathcal{M}\models\phi[x_1,\dots,x_n]\\ &\Leftrightarrow \mathcal{N}\models\phi[f(x_1),\dots,f(x_n)] \qquad (f \text{ is an elementary map})\\ &\Leftrightarrow (f(x_1),\dots,(f(x_n)))\in R^{\mathcal{N}} \qquad \square\end{aligned}$$

For this reason, it is common to use the terms elementary map and elementary embedding interchangeably.

Definition 1.4.3. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures such that $\mathcal{M}$ is a substructure of $\mathcal{N}$. We say that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$, or that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$, denoted by $\mathcal{M}\preceq\mathcal{N}$ if the inclusion map $\iota: M\hookrightarrow N$ is an elementary embedding. This is equivalent to saying that, for any $\mathcal{L}$-formula $\phi(v_1,\dots,v_n)$ and $a_1,\dots,a_n\in M$, then

$$\mathcal{M}\models\phi[a_1,\dots,a_n]\Leftrightarrow\mathcal{N}\models\phi[a_1,\dots,a_n]$$

Example 1.4.4. *Consider the language $\mathcal{L}=\{<\}$. The identity map from $(\mathbb{Q},<)$ into $(\mathbb{R},<)$ is an elementary embedding. In particular, as we will see later, given any dense linearly ordered set without endpoints $(S,<)$, there exists an elementary embedding from $(\mathbb{Q},<)$ into $(S,<)$.*

The definition of an elementary map can be slightly generalized in the following way.

Definition 1.4.5. Let $\mathcal{M},\mathcal{N}$ be two $\mathcal{L}$-structures and $A\subseteq M$ by any subset (not necessarily the domain of a substructure). We say that a map $f: A\to N$ is a *partial elementary* *map* if

$$\mathcal{M} \models \phi[\bar{a}] \Leftrightarrow \mathcal{N} \models \phi[f(\bar{a})]$$

for all $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_n)$ and all $\bar{a} = (a_1, \ldots, a_n) \in A^n$.

We will not use partial elementary maps just now, but they will be useful in later sections.

We are now ready to state and prove the first main result of this section. There are a lot of equivalent ways to state this theorem, however I think that this one is the most straightforward.

Theorem 1.4.6 (Downward Löwenheim-Skolem Theorem). *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $X \subseteq M$ and κ be the cardinality of the set of all $\mathcal{L}$-formulas (i.e. $\kappa = |\mathcal{L}|$). Then, for any cardinal λ such that*

$$|X| + \kappa \leq \lambda \leq |M|$$

there exists an elementary substructure $\mathcal{N}$ of $\mathcal{M}$ with cardinality λ such that $X \subseteq N$.

Proof. Let $N_0 \subset M$ be any subset of cardinality λ such that $X \subseteq N_0$. The main idea of the proof is to gradually add elements to N_0 to form a chain $N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots$ such that $N = \bigcup_{k \in \omega} N_k$ is the universe of the desired elementary substructure. However, we can not add too many elements, or else the cardinality of the final N will be greater than λ. We will build each N_i using induction.

Before we start, we use the well-ordering principle to turn M into a well-ordered set. Later on the proof, when I say "Let x be the least element of M such that ..." I will be referring to this order.

We already have N_0 and assume that we have already built N_k.

For each formula $\phi(x, v_1, \ldots, v_n)$ and $\bar{b} \in N_k^n$ such that

$$\mathcal{M} \models \exists x \phi(x, v_1, \ldots, v_n)[\bar{b}] \tag{1.1}$$

Let $a_{\phi,\bar{b}}$ denote the least element of $\mathcal{M}$ such that $\mathcal{M} \models \phi[a_{\phi,\bar{b}}, \bar{b}]$.

Define

$$N_{k+1} = N_k \cup \{a_{\phi,\bar{b}}, \text{for all formulas } \phi(x, v_1, \ldots, v_n) \text{ and } \bar{b} \in N_k^n \text{ such that (1.1) is true}\}$$

and let $N = \bigcup_{k \in \omega} N_k$. With this done, there are three things we need to check to complete the proof:

1. $|N| = \lambda$

2. N is the universe of some substructure $\mathcal{N}$

3. $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$

1) We start by determining the cardinality of each N_k using induction. We know that $|N_0| = \lambda$. Assume now that $|N_k| = \lambda$. As $N_k \subseteq N_{k+1}$, then $|N_{k+1}| \geq \lambda$. On the other hand, by hypothesis there are $\kappa \leq \lambda$ $\mathcal{L}$-formulas and since the cardinality of the set $\{\bar{b} \in N_k^n : \text{ for some } n\}$ is λ, by forming N_{k+1} we are adding to N_k at most $\kappa\lambda = \lambda$ new elements, meaning that $|N_{k+1}| \leq \lambda + \lambda = \lambda$. So $|N_k + 1| = \lambda$. Now, we have that $N_0 \subseteq N$, meaning that $|N| \geq \lambda$. On the other hand,

$$|N| = \left| \bigcup_{k \in \omega} N_k \right| \leq \aleph_0 \lambda = \lambda$$

So we conclude that $|N| = \lambda$ as desired.

2) For any constant symbol c, consider the formula: $\phi(x)$ given by $x = c$ and take $\bar{b}$ to be empty. Then $\mathcal{M} \models \exists x \phi(x)$ and this is satisfied only by $c^{\mathcal{M}}$. This means that when building N_1, we added $c^{\mathcal{M}}$, and so $c^{\mathcal{M}} \in N$ for all constant symbols c.

Now, let f be any n-ary functional symbol. We will show that $f^{\mathcal{M}}$ is closed in N. Let $\bar{b} = (b_1, \ldots, b_n) \in N^n$ and let k be large enough so that $b_1, \ldots, b_n \in N_k$. Consider the formula $\phi(x, v_1, \ldots, v_n)$ given by $f(v_1, \ldots, v_n) = x$. Then

$$\mathcal{M} \models \exists x \phi(x, v_1, \ldots, v_n)[b_1, \ldots, b_n]$$

which is realized only by $f^{\mathcal{M}}(b_1, \ldots, b_n)$ if M. This means that from $f^{\mathcal{M}}(b_1, \ldots, b_n) \in N_{k+1} \subseteq N$

So, N is the domain of a substructure of $\mathcal{M}$ with $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^n$ for all n-ary relational symbols R, $f^{\mathcal{N}} = f^{\mathcal{M}}|_{N^n}$ for all n-ary functional symbols f and $c^{\mathcal{N}} = c^{\mathcal{M}}$ for all constant symbols c.

3) We will prove this with induction over the construction of formulas. Let $\phi(v_1, \ldots, v_n)$ be an $\mathcal{L}$-formula and $\bar{b} = (b_1, \ldots, b_n) \in N$. Then:

- If ϕ is atomic, $\mathcal{N} \models \phi[\bar{b}] \Leftrightarrow \mathcal{M} \models \phi[\bar{b}]$, as $\mathcal{N}$ is a substructure of $\mathcal{M}$.

- If ϕ is $\neg\psi$ for some ψ, then:

$$\begin{aligned} \mathcal{N} \models \phi[\bar{b}] &\Leftrightarrow \mathcal{N} \not\models \psi[\bar{b}] \\ &\Leftrightarrow \mathcal{M} \not\models \psi[\bar{b}] && \text{(Induction Hypothesis)} \\ &\Leftrightarrow \mathcal{M} \models \phi[\bar{b}] \end{aligned}$$

- If ϕ is $\psi \wedge \chi$, then the proof is analogous just using the induction hypothesis.
- If $\phi = \exists x \psi(x, v_1, \ldots, v_n)$, then if $\mathcal{N} \models \phi[\bar{b}]$, there exists $a \in N$ such that $\mathcal{N} \models \psi[a, \bar{b}]$. By the induction hypothesis, $\mathcal{M} \models \psi(a, \bar{b})$ meaning that $\mathcal{M} \models \phi[\bar{b}]$. On the other hand, let $\mathcal{M} \models \exists x \psi(x, v_1, \ldots, v_n)[\bar{b}]$ and Let k be big enough such that $b_1, \ldots, b_n \in N_k$. Then, by construction, there exists $a \in N_{k+1}$ such that $\mathcal{N} \models \psi[a, \bar{b}]$, meaning that $\mathcal{N} \models \phi[\bar{b}]$. □

As I said earlier, there is another theorem similar to this one, known as Upward Löwenheim-Skolem Theorem. Before proving it, I would like to introduce and prove some useful results about elementary maps and elementary extensions:

Definition 1.4.7. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $\mathcal{L}_M$ be the language obtained from $\mathcal{L}$ by adding new constant symbols for each element of M. We can then expand $\mathcal{M}$ into an $\mathcal{L}_M$-structure by interpreting the new constant symbols as the elements of M they are meant to represent.

The *atomic diagram* of $\mathcal{M}$, denoted by $\text{Diag}_{\text{at}}(\mathcal{M})$ is the set of all atomic $\mathcal{L}_M$-sentences that are true in $\mathcal{M}$.

The *elementary diagram* of $\mathcal{M}$, denoted by $\text{Diag}(\mathcal{M})$ is the set of all $\mathcal{L}_{\mathcal{M}}$-sentences that are true in $\mathcal{M}$

The main purpose of defining the atomic and elementary diagrams of a structure is the following proposition:

Proposition 1.4.8. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\mathcal{N}$ an $\mathcal{L}_M$-structure. Then seeing $\mathcal{N}$ as an $\mathcal{L}$-structure by ignoring the additional constant symbols from $\mathcal{L}_M$, we have that:*

1. *If $\mathcal{N} \models \text{Diag}_{\text{at}}(\mathcal{M})$, then there exists an embedding $j : \mathcal{M} \hookrightarrow \mathcal{N}$;*
2. *If $\mathcal{N} \models \text{Diag}(\mathcal{M})$, then there exists an elementary embedding $j : \mathcal{M} \hookrightarrow \mathcal{N}$.*

Proof. Consider the map $j : \mathcal{M} \to \mathcal{N}$ that maps $m \mapsto m^{\mathcal{N}}$, i.e. it maps each element of M to the interpretation in $\mathcal{N}$ of the corresponding constant symbol. Assume that $\mathcal{N} \models \text{Diag}_{\text{at}}(\mathcal{M})$. We will start by proving that j is a homomorphism.

1) Let R be any n-ary relational symbol from $\mathcal{L}$ and let $m_1, \ldots, m_n$ be elements of M such that $(m_1, \ldots, m_n) \in R^{\mathcal{M}}$. Consider the atomic $\mathcal{L}_M$-sentence ϕ given by $R(m_1, \ldots, m_n)$. As $\mathcal{M} \models \phi$, this means that $\phi \in \text{Diag}_{\text{at}}(\mathcal{M})$ and thus $\mathcal{N} \models \phi$. So we conclude that $(m_1^{\mathcal{N}}, \ldots, m_n^{\mathcal{N}}) \in R^{\mathcal{N}}$ and by the definition of j, this is the same as $(j(m_1), \ldots, j(m_n)) \in R^{\mathcal{N}}$.

2) Now let f be any n-ary functional symbol from $\mathcal{L}$, let $m_1, \ldots, m_n \in M$ and consider the atomic $\mathcal{L}_M$-sentence given by

$$f(m_1, \ldots, m_n) = a$$

where a is the constant symbol associated with the element $f^{\mathcal{M}}(m_1, \ldots, m_n)$. Repeating the same argument, we get that $\mathcal{N} \models \phi$ and thus we conclude that:

$$j(f^{\mathcal{M}}(m_1, \ldots, m_n)) = f^{\mathcal{N}}(j(m_1), \ldots, j(m_n))$$

3) Now, let c be any constant symbol from $\mathcal{L}$, and consider the atomic $\mathcal{L}_M$-formula ϕ given by $c = c^{\mathcal{M}}$. Again, by the same argument, we get that $\mathcal{N} \models \phi$ and thus $c^{\mathcal{N}} = j(c^{\mathcal{M}})$.

Now We will proceed by showing that j is injective. Let $m_1, m_2 \in M$ be two distinct elements of M and consider the atomic $\mathcal{L}_M$-sentence ϕ given by $\neg(m_1 = m_2)$. Then $\mathcal{N} \models \phi$ and thus $j(m_1) \neq j(m_2)$.

In order to conclude that j is an embedding, we only have one more property left to prove. Let R be any n-ary relational symbol and let $m_1, \ldots, m_n \in M$ such that $(j(m_1), \ldots, j(m_n)) \in R^{\mathcal{N}}$. We wish to show that $(m_1, \ldots, m_n) \in R^{\mathcal{M}}$.

If $(m_1, \ldots, m_n) \notin R^{\mathcal{M}}$, then $\mathcal{M} \models \neg R(m_1, \ldots, m_n)$, meaning that $\mathcal{N} \models \neg R(m_1, \ldots, m_n)$ and thus $(j(m_1), \ldots, j(m_n)) \notin R^{\mathcal{N}}$, which is a contradiction.

Now assume that $\mathcal{N} \models \text{Diag}(\mathcal{M})$. Let $\phi(v_1, \ldots, v_n)$ be an $\mathcal{L}$-formula and $m_1, \ldots, m_n \in M$ such that $\mathcal{M} \models \phi[m_1, \ldots, m_n]$. Then $\mathcal{M} \models \phi(m_1, \ldots, m_n)$ and as such $\phi(m_1, \ldots, m_n) \in \text{Diag}(\mathcal{M})$. This means that $\mathcal{N} \models \phi(m_1, \ldots, m_n)$, so we conclude that $\mathcal{N} \models \phi[m_1^{\mathcal{N}}, \ldots, m_n^{\mathcal{N}}]$ which is the same as $\mathcal{N} \models \phi[j(m_1), \ldots, j(m_n)]$.

On the other hand, assume that $\mathcal{N} \models \phi[j(m_1), \ldots, j(m_n)]$ and that $\mathcal{M} \not\models \phi[m_1, \ldots, m_n]$. Then $\mathcal{M} \models \neg\phi(m_1, \ldots, m_n)$. This would imply that $\mathcal{N} \models \neg\phi(m_1, \ldots, m_n)$, meaning that $\mathcal{N} \not\models \phi[j(m_1), \ldots, j(m_n)]$, which is a contradiction. $\square$

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\mathcal{N}$ be an $\mathcal{L}_M$-structure such that $M \subseteq N$ and for each $m \in M$, $m^{\mathcal{N}} = m$. Then the map j we used to prove this proposition is the identity map, meaning that if $\mathcal{N} \models \text{Diag}_{\text{at}}(\mathcal{M})$, then $\mathcal{M}$ is a substructure of $\mathcal{N}$, and if $\mathcal{N} \models \text{Diag}(\mathcal{M})$, $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.

However, if this is not the case, we still have an isomorphism $\mathcal{M} \simeq j(\mathcal{M})$, and by making this identification, we can say that if $\mathcal{N} \models \text{Diag}_{\text{at}}(\mathcal{M})$, then $\mathcal{M}$ is a substructure of $\mathcal{N}$ and if $\mathcal{N} \models \text{Diag}(\mathcal{M})$, then $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$. This is justified by the fact that if $\mathcal{N} \models \text{Diag}(\mathcal{M})$, knowing how the elements of $j(\mathcal{M})$ and $\mathcal{N}$ behave allows us to build an extension $\mathcal{M} \preceq \mathcal{N}'$ with $\mathcal{N}' \simeq \mathcal{N}$.

The following proposition is sometimes useful as it gives us an equivalent condition for a substructure to be an elementary substructure, furthermore, we have already proved it implicitly when proving the Downward Löwenheim-Skolem Theorem. I only state this proposition for the sake of sake of completion, but we will not use it in this work.

Proposition 1.4.9 (Tarski-Vaught Test). *Let $\mathcal{M}$ be a structure and $\mathcal{N}$ a substructure. Then the following are equivalent:*

- *$\mathcal{N}$ is an elementary substructure of $\mathcal{M}$;*
- *For any formula $\phi(x, v_1, \ldots, v_n)$ and any $\bar{b} \in N^n$, if $\mathcal{M} \models \exists x \phi(x, \bar{v})[\bar{b}]$ then there exists $a \in N$ such that $\mathcal{N} \models \phi[a, \bar{b}]$.*

Now, for the other main theorem of this section:

Theorem 1.4.10 (Upward Löwenheim-Skolem Theorem). *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and let κ be the cardinality of the set of all $\mathcal{L}$-formulas (i.e. $\kappa = |\mathcal{L}|$). Then, for any cardinal λ such that:*

$$|M| + \kappa \leq \lambda$$

there exists a proper elementary extension of $\mathcal{M}$ with cardinality λ.

Proof. We begin by considering a new language $\mathcal{L}_M^*$ obtained from $\mathcal{L}_M$ by adding the new constant symbols c_α for each ordinal $\alpha < \lambda$. Let T be the $\mathcal{L}_M^*$-theory given by:

$$T = \bigcup_{m \in M \alpha < \lambda} \{m \neq \alpha\} \cup \bigcup_{\alpha, \beta \leq \lambda \alpha \neq \beta} \{c_\alpha \neq c_\beta\}$$

Now, consider the $\mathcal{L}_M^*$-theory given by $\text{Diag}(\mathcal{M}) \cup T$. As $\mathcal{M}$ is infinite, every finite subset of $\text{Diag}(\mathcal{M}) \cup T$ is satisfied by $\mathcal{M}$ by choosing appropriate interpretations for the new constants. By the compactness theorem, there exists $\mathcal{N}_0 \models \text{Diag}(\mathcal{M}) \cup T$. Now, as $\mathcal{N}_0 \models \text{Diag}(\mathcal{M})$, we know that $\mathcal{N}_0$ is an elementary extension of $\mathcal{M}$, and because $\mathcal{N}_0 \models T$, we know that $|N_0| \geq \lambda$.

Now, as $|M| + |\mathcal{L}_0^*| \leq \lambda \leq |N_0|$, by the Downward Löwenheim-Skolem Theorem there exists an elementary substructure $\mathcal{N} \preceq \mathcal{N}_0$ such that $M \subseteq N$ (meaning that $\mathcal{M} \leq \mathcal{N}$) and $|N| = \lambda$. We claim that $\mathcal{N}$ is an elementary extension of $\mathcal{M}$. To see this, let $\phi(v_1, \ldots, v_n)$ be any $\mathcal{L}$-formula and let $\bar{a} \in M^n$. Then $\mathcal{M} \models \phi[\bar{a}] \Leftrightarrow \mathcal{N}_0 \models \phi[\bar{a}] \Leftrightarrow \mathcal{N} \models \phi[\bar{a}]$. □

We will end this section by proving just one fact about elementary extensions that will be useful later:

Definition 1.4.11. Let $(I, <)$ be a linearly ordered set and for each $i \in I$ let $\mathcal{M}_i$ be an $\mathcal{L}$-structure. We say that $(\mathcal{M}_i : i \in I)$ is a chain of $\mathcal{L}$-structures if $\mathcal{M}_i \leq \mathcal{M}_j$ for any $i < j$. If $\mathcal{M}_i \preceq \mathcal{M}_j$ for any $i < j$, we say that $(\mathcal{M}_i : i \in I)$ is an elementary chain .

For a chain $(\mathcal{M}_i : i \in I)$, we can define $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ as follows:

- The universe of M is $\bigcup_{i \in I} M_i$;
- Let c be any constant symbol. For any $i, j \in I$. we have $c^{\mathcal{M}_i} = c^{\mathcal{M}_j}$, so we define $c^{\mathcal{M}} = c^{\mathcal{M}_i}$, for all $i \in I$;
- Let f be an n-ary functional symbol and let $\bar{a} \in M^n$. There exists $k \in I$ such that $\bar{a} \in M_k^n$, and for any $i > k$, $f^{\mathcal{M}_i}(\bar{a}) = f^{\mathcal{M}_k}(\bar{a})$. So $f^{\mathcal{M}} = \bigcup_{i \in I} f^{\mathcal{M}_i}$ is a well-defined function.
- Let R be any n-ary relational symbol and let $\bar{a} \in M^n$. By the same argument, there exists $k \in I$ such that for all $i > k$, $\bar{a} \in R^{\mathcal{M}_i}$ if and only if $\bar{a} \in R^{\mathcal{M}_k}$. So we define $R^{\mathcal{M}} = \bigcup_{i \in I} R^{\mathcal{M}_i}$

Proposition 1.4.12. *Let $(\mathcal{M}_i : i \in I)$ be an elementary chain. Then, $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ is an elementary extension of any $\mathcal{M}_i$.*

Proof. Let $i \in I$, let $\phi(v_1, \ldots, v_n)$ be an $\mathcal{L}$-formula and $\bar{a} \in M_i^n$. We will use induction over the complexity of ϕ. If ϕ is atomic, because $\mathcal{M}_i$ is a substructure of $\mathcal{M}$, we have $\mathcal{M}_i \models \phi[\bar{a}]$ if and only if $\mathcal{M} \models \phi[\bar{a}]$.

If ϕ is $\neg\psi$ or $\psi \wedge \chi$, the result follows immediately from the induction hypothesis.

Now, if ϕ is $\exists x\psi(x, v_1, \ldots, v_n)$ and $\mathcal{M}_i \models \phi[\bar{a}]$, then $\mathcal{M}_i \models \psi[b, \bar{a}]$ for some $b \in M_i$. By the induction hypothesis, $\mathcal{M} \models \psi[b, \bar{a}]$ and thus $\mathcal{M} \models \phi[\bar{a}]$. On the other hand, if $\mathcal{M} \models \phi[\bar{a}]$, then $\mathcal{M} \models \psi[b, \bar{a}]$ for some $b \in M$. If $b \in M_i$, then $\mathcal{M}_i \models \phi[\bar{a}]$. However, if $b \notin M_i$, there exists $j > i$ such that $b \in M_j$. Because $\mathcal{M}_i \preceq \mathcal{M}_j$, and $\mathcal{M}_j \models \psi[b, \bar{a}]$ we have that $\mathcal{M}_i \models \psi[b, \bar{a}]$ and thus $\mathcal{M}_i \models \phi[\bar{a}]$. □

1.5. Complete and Categorical Theories

In this section we will dive deeper into the theory of complete theories. Now that we have more model theoretic tools at our disposal then we did when we first defined what it means for a theory to be complete back in Definition 1.1.33, we will explore some interesting consequences of completeness.

Before continuing, recall a satisfiable theory is complete if any two models of such theory are elementary equivalent (Definition 1.1.33). For example, as proved earlier, for any structure $\mathcal{M}$, the theory $\text{Th}(\mathcal{M})$ is complete.

Note that in a complete theory T, if $\mathcal{M} \models T$, then $\mathcal{M} \models \phi$ if and only if $T \models \phi$.

The following gives an alternative characterization of complete theories.

Proposition 1.5.1. *Let T be an $\mathcal{L}$-theory. Then T is complete if and only if, for every $\mathcal{L}$-sentence ϕ, either $T \models \phi$ or $T \models \neg\phi$.*

Proof. Assume that T is complete, let $\mathcal{M} \models T$ and let ϕ be an $\mathcal{L}$-sentence. We know that either $\mathcal{M} \models \phi$ or $\mathcal{M} \models \neg\phi$, meaning that either $T \models \phi$ or $T \models \neg\phi$.

On the other hand, assume that for all ϕ, either $T \models \phi$ or $T \models \neg\phi$ and let $\mathcal{M}, \mathcal{N} \models T$. Let $\phi \in \text{Th}(\mathcal{M})$. If $T \models \neg\phi$, then we would have that $\mathcal{M} \models \neg\phi$, which is impossible. So $T \models \phi$, which implies that $\mathcal{N} \models \phi$, and thus $\mathcal{N} \models \text{Th}(\mathcal{M})$. By the completeness of $\text{Th}(\mathcal{M})$, $\mathcal{M} \equiv \mathcal{N}$. □

Now, I will introduce another class of theories, namely, categorical theories, which forms a very important and interesting class of theories with deep model-theoretic properties.

Definition 1.5.2. Let T be an $\mathcal{L}$-theory and κ an infinite cardinal. We say that T is κ-*categorical* if T has at least one model of cardinality κ and all models of cardinality κ are isomorphic.

Categoricity at first sight seems totally unrelated to completeness, however, using the Löwenheim-Skolem Theorem, we have the following useful tool to prove that theories are complete.

Theorem 1.5.3 (Vaught's Test). *Let T be a satisfiable $\mathcal{L}$-theory with no finite models. If T is κ-categorical, for some $k \geq \max\{|\mathcal{L}|, \aleph_0\}$, then T is complete.*

Proof. Let $\mathcal{M}, \mathcal{N} \models T$. The main idea is to use Löwenheim-Skolem to find elementary extensions or substructures of $\mathcal{M}$ and $\mathcal{N}$ with cardinality κ. We will divide the proof into 4 cases:

1. If $|M|, |N| \leq \kappa$, then by the Upward Löwenheim-Skolem there are elementary extensions $\mathcal{M} \preceq \mathcal{M}'$ and $\mathcal{N} \preceq \mathcal{N}'$ with $|M'| = |N'| = \kappa$. By categoricity $\mathcal{M}' \simeq \mathcal{N}'$, so that $\mathcal{M} \equiv \mathcal{M}' \simeq \mathcal{N}' \equiv \mathcal{N}$.
2. If $\kappa \leq |M|, |N|$, we do the same thing but using the downward Löwenheim-Skolem theorem.
3. If $|M| \leq \kappa \leq |N|$ we use the Upward Löwenheim-Skolem on $\mathcal{M}$ and the Downward Löwenheim-Skolem on $\mathcal{N}$.
4. If $|N| \leq \kappa \leq |M|$ we use the Upward Löwenheim-Skolem on $\mathcal{N}$ and the Downward Löwenheim-Skolem on $\mathcal{M}$. □

We will now use Vaught's test to give some examples of complete theories, starting with the theory of dense linear orders without endpoints. Recall that a linearly ordered set $(I, <)$ is dense if, for all $x, y \in I$ with $x < y$, there exists $c \in I$ such that $x < c < y$. We say that $(I, <)$ is a dense linear order without endpoints (DLO) if $(I, <)$ is a dense linear order and, for all $x \in I$, there are a and b such that $x < a$ and $b < x$. We use these properties as axioms to define the theory of dense linear orders without endpoints in the language $\mathcal{L}$ consisting of only one relational binary symbol $<$:

$$\begin{aligned} DLO := \{&\forall x \neg(x < x), \\ &\forall x \forall y (x < y \rightarrow \neg(y < x)), \\ &\forall x \forall y \forall z [(x < y \land y < z) \rightarrow x < z], \\ &\forall x \forall y [x \neq y \rightarrow (x < y \lor y < x)], \\ &\forall x \forall y [x < y \rightarrow (\exists z (x < z \land z < y))], \\ &\forall x \exists a \exists b (a < x \land x < b)\} \end{aligned}$$

So $(I,<) \models DLO$ if and only if $(I,<)$ is a dense linear order without endpoints. The following theorem is due to Georg Cantor:

Theorem 1.5.4 (Cantor's isomorphism theorem)**.** *Any two countable dense linear orders without endpoints are isomorphic.*

Proof. Let $(A,<)$ and $(B,\prec)$ be two DLO, with $A = (a_1, a_2, \ldots)$ and $B = (b_1, b_2, \ldots)$. When we refer to the earliest element that satisfies some property, we will be referring to this ordering of the elements of A and B.

We will build an isomorphism $f : A \to B$ using a method known as "back-and-forth", which is fairly used in model theory to build isomorphism or elementary embeddings between two structures. The main idea of the method is the following: We build a sequence of subsets $A_1 \subseteq A_2 \subseteq \ldots$ of A, $B_1 \subseteq B_2 \subseteq \ldots$ of B and a sequence of functions $f_i : A_i \to B_i$ such that $A = \bigcup_{i\in\omega} A_i$, $B = \bigcup_{i\in\omega} A_i$ and $f_1 \subseteq f_2 \subseteq \ldots$, using induction. At the end of the induction, the function $f = \bigcup_{i\in\omega} f_i$ will be our desired isomorphism. We usually do this via induction.

Stage 0) Let $A_1 = \{c_1\}$, $B_1 = \{d_1\}$ and $f_1 = \{(c_1, d_1)\}$, where $c_1 = a_1$ and $d_1 = b_1$.

Stage $k+1$ when k is even) In this step, we will make sure that in the end $A = \bigcup_{i\in\omega} A_i$. Let $A_{k+1} = A_k \cup \{c_{k+1}\}$, $B_{k+1} = B_k \cup \{d_{k+1}\}$, and $f_{k+1} = f_k \cup \{(c_{k+1}, d_{k+1})\}$, where c_{k+1} is the earliest element from A that we have not added to A_k, and d_{k+1} is the earliest element from B such that f_{k+1} is an order preserving map from $(A_{k+1},<)$ to $(B_{k+1},\prec)$.

Stage $k+1$ when k is odd) In this step, we will make sure that $B = \bigcup_{i\in\omega} B_i$ by doing the same we did in the previous step. Let $B_{k+1} = B_k \cup \{d_{k+1}\}$, $A_{k+1} = A_k \cup \{c_{k+1}\}$ and $f_{k+1} = f_k \cup \{(c_{k+1}, d_{k+1})\}$, where d_{k+1} is the earliest element from B that we have not added to B_k, and c_{k+1} is the earliest element from A such that f_{k+1} is an order preserving map from $(A_{k+1},<)$ to $(B_{k+1},\prec)$.

There is one thing we still need to check. If k is even or odd, how can we make sure that there exists a pair (c_{k+1}, d_{k+1}) that satisfies all the conditions we want? Since both cases are symmetric, We will see that such a pair always exists for the step where k is even.

Because A is infinite and A_k is finite, there exists the earliest element c_{k+1} of A that is not in A_k. Now we just need to prove that there exists $b \in B \setminus B_k$ such that, for all $\alpha \in A_k$:

$$\alpha < c_{k+1} \Leftrightarrow f_k(\alpha) \prec b \tag{1.2}$$

Let σ be a permutation such that the elements of A_k are ordered as such: $c_{\sigma(1)} < c_{\sigma(2)} < \ldots < c_{\sigma(k)}$ and the elements of B_k as such: $d_{\sigma(1)} < d_{\sigma(2)} < \ldots < d_{\sigma(k)}$.

This divides A into $k+1$ intervals, namely: $(-\infty, c_{\sigma(1)}), (c_{\sigma(1)}, c_{\sigma(2)}), \ldots, (c_{\sigma(k)}, \infty)$, and the same happens with B, where $(-\infty, c_{\sigma(1)})$ denotes the set of all x such that $x < c_{\sigma(1)}$ and $(c_{\sigma(k)}, \infty)$ is defined analogously. Now, we have some different cases to consider:

1) If $c_{k+1} \in (c_{\sigma(k)}, \infty)$, then we can pick any $b \in (c_{\sigma(k)}, \infty)$, which exists because $(B, \prec)$ does not have endpoints;

2) If $c_{k+1} \in (-\infty, c_{\sigma(1)})$, then we can pick any $b \in (-\infty, d_{\sigma(1)})$, which exists for the same reason;

3) If If $c_{k+1} \in (c_{\sigma(j)}, c_{\sigma(j+1)})$, then we can pick any $b \in (d_{\sigma(j)}, d_{\sigma(j+1)})$, which exists because $(B, \prec)$ is dense.

This choice of b will always satisfy (1.2) because, by the induction hypothesis, f_k is order-preserving. So there exists an earliest element d_{k+1} that satisfies (1.2).

Now let $f = \bigcup_{i \in \omega} f_i$. Let $x, y \in A$ such that $x < y$. There exists $k \in \mathbb{N}$ such that $x, y \in A_k$ and this $f_k(x) < f_k(y)$, meaning that $f(x) < f(y)$. $\square$

We can re-state this theorem as: DLO is $\aleph_0$-categorical, and thus by Vaught's test we have the following.

Corollary 1.5.5. *DLO is a complete theory.*

Some more examples of categorical theories include:

- The theory of torsion-free divisible abelian groups is κ-categorical for all $\kappa > \aleph_0$;
- ACF_p for $p = 0$ or p prime is κ-categorical for all $\kappa > \aleph_0$;
- The theory of vector spaces over a countable field is κ-categorical for any $\kappa > \aleph_0$.

For more details see Chapter 2.2 in [5]

In particular, ACF_p is complete, for $p = 0$ or p prime, as there is no finite algebraically closed field.

Before we conclude this chapter, for the sake of completion, I will just show one interesting application of the completeness of ACF_p outside of model theory.

Theorem 1.5.6 (Lefschetz Principle). *Let ϕ be an $\mathcal{L}_r$-sentence in the language of rings. Then the following are equivalent:*

1. *ϕ is true in $\mathbb{C}$;*
2. *ϕ is true in every ACF of characteristic 0;*
3. *ϕ is true in some ACF of characteristic 0;*
4. *There are arbitrarily large primes p such that ϕ is true in some ACF of characteristic p;*
5. *There is an m such that for all primes $p > m$, ϕ is true in all ACF of characteristic p.*

Proof. By completeness of ACF_0, statements (1),(2) and (3) are equivalent. Moreover, (5) implies (4) trivially.

$(2) \Rightarrow (5)$: If $ACF_0 \models \phi$, then by Proposition 1.3.13, there exists a finite subset $\Delta \subseteq ACF_0$ such that $\Delta \models \phi$. From our definition of ACF_0 in example 1.1.30, there exists a prime p such that for any prime $q > p, ACF_q \models \Delta$ and thus $ACF_q \models \phi$.

$(4) \Rightarrow (3)$: Consider the theory $ACF_0 \cup \{\phi\}$. For any finite subset $\Delta \subseteq ACF_0 \cup \{\phi\}$, there exist a prime p and $F \models ACF_p$ such that $F \models \Delta$. By compactness, there exists $R \models ACF_0 \cup \{\phi\}$. □

From this follows the following theorem:

Corollary 1.5.7 (Ax–Grothendieck theorem). *Every injective polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ is surjective.*

Proof. I will only sketch the proof, as aside from the Lefschetz Principle, the rest of the proof is mainly using concepts from field theory that are outside the scope of model theory.

Fix $n \in \mathbb{N}$, let $p \in \mathbb{N}$ be prime and note that any injective polynomial map $\mathbb{Z}_p^n \to \mathbb{Z}_p^n$ is also subjective as $\mathbb{Z}_p^n$ is finite. Using field theory, from this one can prove that any injective polynomial map $(\bar{\mathbb{Z}}_p)^n \to (\bar{\mathbb{Z}}_p)^n$ is subjective, where $\bar{\cdot}$ denotes the algebraic closure.

Given a field $\mathbb{F}$ and $d \in \mathbb{N}$, it is possible to write a sentence ϕ_d in the language of rings such that $\mathbb{F} \models \phi_d$ if and only if every injective polynomial map $\mathbb{F}^n \to \mathbb{F}^n$ of maximum degree d is surjective. As we just saw, $\bar{\mathbb{Z}}_p \models \phi_d$, for all $d \in \mathbb{N}$. As p is an arbitrary prime, by the Lefschetz Principle $\mathbb{C} \models \phi_d$ for any $d \in \mathbb{N}$.

For the full proof see Theorem 2.2.11 in [5]. □

1.6. Quantifier Elimination

In some cases, a formula with quantifiers can be equivalent to a quantifier-free formula. For example, take the formula $\phi(a,b,c)$ given by $\exists x(ax^2+bx+c=0)$. In $\mathbb{R}$, the equation $ax^2+bx+c=0$ has a solution if $a \neq 0$ and $b^2-4ac \geq 0$ or if $a=0$ and either $b \neq 0$ or $c=0$. So consider the formula $\psi(a,b,c)$ given by $(a \neq 0 \wedge b^2-4ac \geq 0) \vee (a=0 \wedge (b \neq 0 \vee c=0))$. Then for any $a,b,c \in \mathbb{R}$, we have $\mathbb{R} \models \phi[a,b,c]$ if and only if $\mathbb{R} \models \psi[a,b,c]$. It is easy to see why this is a desirable behavior: quantifiers introduce a lot of complexity when studying, among other things, definable sets, and being able to find a quantifier-free formula equivalent to any formula is a very powerful tool.

This motivates the following definition.

Definition 1.6.1. Let T be an $\mathcal{L}$-theory. We say that T has *quantifier elimination* (QE) if, for every $\mathcal{L}$-formula $\phi(v_1,\ldots,v_n)$, there exists a quantifier-free $\mathcal{L}$-formula $\psi(v_1,\ldots,v_n)$ such that

$$T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$$

i.e. for all models $\mathcal{M} \models T$, and for all $\bar{a} \in M^n$, $\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M} \models \psi[\bar{a}]$.

Before looking at some examples, we will fist build a simple tool that allows us to prove that a theory has quantifier elimination.

We say that a formula is a basic conjunction if it is a conjunction of literals.

Proposition 1.6.2. *Let T be an $\mathcal{L}$-theory. Suppose that, given any basic conjunction $\phi(v_1,\ldots,v_n,y)$, there exists a quantifier-free formula $\psi(v_1,\ldots,v_n)$ such that, for every model $\mathcal{M}$ of T and every $\bar{a} \in M^n$,*

$$\mathcal{M} \models \exists y \phi[\bar{a}] \Leftrightarrow \mathcal{M} \models \psi[\bar{a}]$$

Then T has quantifier elimination.

Proof. To simplify the notation on this proof, let $\phi \sim \psi$ denote that the formulas ϕ and ψ are equivalent.

We will use induction over the definition of formula to show that T has quantifier elimination.

If ϕ is atomic, then by definition ϕ is already quantifier-free.

If $\phi = \neg\psi$ and there exists a quantifier-free formula ψ_0 equivalent to ψ, then ϕ is equivalent to $\neg\psi_0$ which is quantifier-free.

Finally, assume that $\phi = \exists y\psi(v_1, \ldots, v_n, y)$ and assume that there exists a quantifier-free formula $\psi_0(v_1, \ldots, v_n, y)$ such that $\psi(v_1, \ldots, v_n, y) \sim \psi_0(v_1, \ldots, v_n, y)$. By Proposition 1.1.24, there are basic conjunctions $\theta_1(v_1, \ldots, v_n, y), \ldots, \theta_k(v_1, \ldots, v_n, y)$ such that

$$\psi_0(v_1, \ldots, v_n, y) \sim \bigvee_{i=1}^{k} \theta_i(v_1, \ldots, v_n, y)$$

In particular, this implies that

$$\begin{aligned}\phi &\sim \exists y \bigvee_{i=1}^{k} \theta_i(v_1, \ldots, v_n, y) \\ &\sim \bigvee_{i=1}^{k} \exists_y \theta_i(v_1, \ldots, v_n, y)\end{aligned}$$

As each θ_i is a literal, by our hypothesis, there exists a quantifier-free formula $\theta_i'(v_1, \ldots, v_n)$, such that $\exists y\theta_i(v_1, \ldots, v_n, y) \sim \theta_i'(v_1, \ldots, v_n)$, for each $i = 1, \ldots, k$, meaning that

$$\phi \sim \bigvee_{i=1}^{k} \theta_i'(v_1, \ldots, v_n)$$

which is quantifier-free. □

I will demonstrate with an example how this criteria can be used to prove that a theory has quantifier elimination.

Proposition 1.6.3. *DLO has quantifier elimination.*

Proof. In order to use Proposition 1.6.2, we first need to understand the structure of the basic conjunctions. Recall that DLO is a theory in the language $\mathcal{L} = \{<\}$ where $<$ is a binary relational symbol. As there are no functional nor constant symbols, any term $\tau(v_1, \ldots, v_n)$ is simply equal to v_i for some $i = 1, \ldots, n$.

With this in mind, the atomic formulas are either $v_i = v_j$ or $v_i < v_j$ for variables v_i, v_j. In particular, any literal is of the form $v_i = v_j$, $v_i < v_j$, $\neg(v_i = v_j)$ or $\neg(v_i < v_j)$ for variables v_i, v_j.

Let $\phi(v_1, \ldots, v_n, y)$ be a basic conjunction and write

$$\exists y\phi = \exists y \bigwedge_{i=1}^{k} \theta_i(v_1, \ldots, v_n, y)$$

Where each θ_i is a literal. We wish to show that $\exists y\phi$ is equivalent to a quantifier-free formula, which by Proposition 1.6.2, is enough to conclude that DLO has quantifier elimination. Start by noting that if y does not occur in some literal $\theta_i(v_1, \ldots, v_n, y)$, then we can move it outside the scope of $\exists y$, and after doing that for every literal where y does not occur we are only left under the scope of $\exists y$ with literals where y does occur, i.e. without loss of generality, we may assume that y occurs in each $\theta_i(v_1, \ldots, v_n, y)$. Now, note that if some literal $\theta_i(v_1, \ldots, v_n, y)$ is a contradiction, for example, $v_i \neq v_i$ or $y < y$, then $\exists y\phi$ is never true in any model of DLO, meaning that $\exists y\phi$ would be equivalent to the quantifier-free formula $v_1 \neq v_1$. This means that, without loss of generality, we may assume that no θ_i is a contradiction. With all this assumptions in place, we have that

$$\exists y\phi = \exists y \Big[\bigwedge_{i \in I}(y = v_i) \wedge \bigwedge_{j \in J}(y < v_j) \wedge \bigwedge_{k \in K}(v_k < y) \wedge \\ \bigwedge_{i' \in I'} \neg(y = v_{i'}) \wedge \bigwedge_{j' \in J'} \neg(y < v_{j'}) \wedge \bigwedge_{k' \in K'} \neg(v_{k'} < y) \Big]$$

For finite subsets I, J, K, I', J', K' of $\{1, \ldots, n\}$. Note however that we can further simplify this by considering the fact that:

- $\neg(y = v_{i'})$ is equivalent to $y < v_{i'} \vee v_{i'} < y$;
- $\neg(y < v_{j'})$ is equivalent to $y = v_{j'} \vee v_{i'} < y$;
- $\neg(v_{k'} < y)$ is equivalent to $y = v_{k'} \vee y < v_{k'}$.

And as $\exists$ commutes with $\vee$, we can assume without loss of generality that

$$\exists y\phi = \exists y \left[\bigwedge_{i \in I}(y = v_i) \wedge \bigwedge_{j \in J}(y < v_j) \wedge \bigwedge_{k \in K}(v_k < y) \right]$$

For finite subsets I, J, K of $\{1, \ldots, n\}$.

To finish this proof, we divide it into two cases:

Case 1: $I \neq \emptyset$.

Let $t \in \{v_i : i \in I\}$ be a variable, and let

$$\psi(v_1, \ldots, v_n) = \bigwedge_{i \in I} t = v_i \wedge \bigwedge_{j \in J} t < v_j \wedge \bigwedge_{k \in K} v_k < t$$

Then, for any $\mathcal{M} \models DLO$ and $\bar{a} \in M^n$, we have that $\mathcal{M} \models \phi[\bar{a}] \Leftrightarrow \mathcal{M} \models \psi[\bar{a}]$

Case 2: $I = \emptyset$.

Consider the formula $\psi(v_1, \ldots, v_n)$ given by

$$\psi(v_1, \ldots, v_n) = \begin{cases} v_1 = v_1 & \text{if } J = \emptyset \text{ or } K = \emptyset \\ \bigwedge_{j \in J, k \in K} (v_k < v_j) & \text{otherwise} \end{cases}$$

Then, given any $\mathcal{M} \models DLO$ and $\bar{a} \in M^n$, as $\mathcal{M}$ is densely ordered without endpoints, we have that $\mathcal{M} \models \phi[\bar{a}] \Leftrightarrow \mathcal{M} \models \psi[\bar{a}]$. □

A similar application of Proposition 1.6.2 proves that, for example, the theory of infinite sets in the language $\mathcal{L} = \emptyset$ has quantifier elimination. However, most of the time, to prove that a theory has quantifier elimination, more sophisticated methods are necessary.

Some other examples of theories with quantifier elimination include:

- Theory of torsion-free divisible abelian groups;
- Theory of ordered divisible abelian groups;
- Theory of real closed fields on $\mathcal{L}_{or}$;
- ACF.

For more details about these examples and about quantifier elimination in general, see Chapter 3.1 in [5].

1.7. Types

In this section, we will briefly discuss and prove some results about types. Types are an incredibly powerful and useful model-theoretic construction, so much so that, as one starts to explore more advanced model-theoretic techniques, understanding and being comfortable using types is imperative. Intuitively, we can think of a type over some structure $\mathcal{M}$ as being an object that describes some element that could exist on $\mathcal{M}$, or in other words, an object that describes an element whose existence in $\mathcal{M}$ would not give rise to any contradictions. This will become more precise and make more sense when we rigorously define what a type is and prove some results about them. The two main theorems in this section are the realizing types theorem and the omitting types theorem. Thinking again in terms of the analogy I used earlier, the realizing types theorem tells us that given a type in some structure $\mathcal{M}$, we can always expand our structure to include the element that the type describes. The omitting types theorem is the dual of this, as it answers the question: given a type, is it always possible to find a structure where the element represented by that type does not exist?

1.7.1 The Realizing Types Theorem

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. Consider the language $\mathcal{L}_A$ obtained by adding a new constant symbol for each element of A. We can naturally turn $\mathcal{M}$ into an $\mathcal{L}_A$-structure by interpreting the new constant symbols as the elements of A they are meant to represent. Let $\text{Th}_A(\mathcal{M})$ denote the full theory of $\mathcal{M}$ in the language $\mathcal{L}_A$.

Formally, a type is defined in the following way:

Definition 1.7.1. Let p be a set of $\mathcal{L}_A$-formulas with free variables among $v_1, \ldots, v_n$. Then we say that p is an n-*type over* A (or simply n-type when the set A is implicit or arbitrary) if $p \cup \text{Th}_A(\mathcal{M})$ is satisfiable. We sometimes write $p(v_1, \ldots, v_n)$ to emphasize the fact that p in an n-type.

Note that until now, we have only used the term ”satisfiable” when dealing with sets of sentences, however the set p in Definition 1.7.1 has formulas with free variables. We generalize the definition of a satisfiable theory as follows: Let Γ be a set of $\mathcal{L}$-formulas with free variables from $v_1, \ldots, v_n$. We say that Γ is satisfiable if there exists an $\mathcal{L}$-structure $\mathcal{M}$ and an n-tuple $\bar{a} \in M^n$ such that, for all $\phi(v_1, \ldots, v_n) \in \Gamma$ we have that $\mathcal{M} \models \phi[\bar{a}]$.

Example 1.7.2. *Consider the structure $\mathcal{M} = (\mathbb{Z}, <, 0)$, and let*

- $p(v) = \{v < 0\}$;
- $q(v) = \{v > 0, v > 1, v > 2, \ldots\}$.

Note that p is trivially satisfiable in $\mathcal{M}$ by interpreting v as any negative integer, so $q(v)$ is a 1-type (over $\emptyset$).

On the other hand, no element of $\mathcal{M}$ satisfies $q \cup Th_{\mathbb{N}}(\mathcal{M})$. However, note that any finite subset $\Delta \subseteq q \cup Th_{\mathbb{N}}(\mathcal{M})$ is satisfiable by interpreting v as sufficiently large integer and thus by compactness, $q \cup Th_{\mathbb{N}}(\mathcal{M})$ is satisfiable, meaning that $q(v)$ is indeed a 1-type (over $\mathbb{N}$).

Referring back to the analogy at the start of this section, we can think of the type q as describing an infinite element with respect to the other $<$ in $\mathbb{Z}$. Of course that no such element exists in $\mathbb{Z}$, however, since $q \cup Th_{\mathbb{N}}(\mathcal{M})$ is satisfiable, the existence of such an infinite element would not give rise to any logical contradictions.

The fundamental difference between the two types given in Example 1.7.2 is that p is satisfiable in $\mathcal{M}$ itself, while for q this is not the case. This motivates the following definition.

Definition 1.7.3. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ and p be an n-type over A. We say that p is *realized* in $\mathcal{M}$ if there exists $\bar{a} \in M^n$ such that for all $\phi \in p$ we have $\mathcal{M} \models \phi[\bar{a}]$. If this is the case, we also say that $\bar{a}$ realizes p. If p is not realized in $\mathcal{M}$, then we say that $\mathcal{M}$ *omits* p.

One particularly important kind of types are complete types, as we will see later in this section. Complete types are types which are maximal with respect to inclusion. Equivalently we have the following definition:

Definition 1.7.4. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ and p be an n-type over A. We say that p is *complete* if for any $\mathcal{L}_A$-formula ϕ with free variables from $v_1, \ldots, v_n$ either $\phi \in p$ or $\neg\phi \in p$. If p is not a complete type we say that p is an *incomplete* or *partial* type. We denote the set of all complete n-types over A as $S_n^{\mathcal{M}}(A)$.

Note that if $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ and $A \subseteq M$, then $S_n^{\mathcal{M}}(A) = S_n^{\mathcal{N}}(A)$.

Now, given any $\bar{a} \in M^n$ and $A \subseteq M$, consider the set of $\mathcal{L}_A$-formulas:

$$\{\phi(v_1, \ldots, v_n) : \mathcal{M} \models \phi[\bar{a}]\}$$

Because for any $\mathcal{L}_A$-formula $\phi(v_1, \ldots, v_n)$, either $\mathcal{M} \models \phi[\bar{a}]$ or $\mathcal{M} \not\models \phi[\bar{a}]$, this is a complete type, which we denote as $\mathrm{tp}^{\mathcal{M}}(\bar{a}/A)$, or simply as $\mathrm{tp}(\bar{a}/A)$ when $\mathcal{M}$ is obvious from context. If A is the empty set, we simply write $\mathrm{tp}^{\mathcal{M}}(\bar{a})$ instead of $\mathrm{tp}^{\mathcal{M}}(\bar{a}/\emptyset)$. Not all complete types are of this form, however as we will see later, complete types are closely related to these kinds of types.

Theorem 1.7.5 (Realizing types). *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ and p by any n-type over A. There exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ that realizes p.*

Proof. It suffices to show that the set of $\mathcal{L}_M$-formulas $\Gamma = p \cup \text{Diag}(\mathcal{M})$ is satisfiable by Proposition 1.4.8.

In order to show that Γ is satisfiable, we would like to use the compactness theorem, however as the compactness theorem only applies to sets of sentences, we need to make some adjustments to Γ first.

Let $\mathcal{L}_M^* \supseteq \mathcal{L}_M$ be a new language obtained from $\mathcal{L}_M$ by adding new constant symbols $c_1, \dots, c_n$ and let

$$p' = \{\phi(c_1, \dots, c_n) : \phi(v_1, \dots, v_n) \in p\}$$

be the set of $\mathcal{L}_M^*$-sentences obtained from substituting each free occurrence of v_i by c_i in each formula of p, and let $\Gamma' = p' \cup \text{Diag}(\mathcal{M})$.

Let $\Delta \subseteq \Gamma'$ be any finite subset. We start by building a $\mathcal{L}_M^*$-structure that models Δ.

Let $a_1, \dots, a_i \in A$ be the constants from A that occur in any sentence of Δ and let $b_1, \dots, b_l \in M \setminus A$ be the constants that do not originate from A yet appear in any formula of $\Delta \cap \text{Diag}(\mathcal{M})$.

So we have that

$$\Delta = \{\phi_1(\bar{c}, \bar{a}), \dots, \phi_m(\bar{c}, \bar{a})\} \cup \{\psi_1(\bar{a}, \bar{b}), \dots, \psi_k(\bar{a}, \bar{b})\}$$

for some $\phi_1, \dots, \phi_m \in p'$ and $\psi_1, \dots, \psi_k \in \text{Diag}(\mathcal{M})$.

Let $\mathcal{N}_0$ be an $\mathcal{L}_A$-structure such that $\mathcal{N}_0 \models p \cup \text{Th}_A(\mathcal{M})$, which exists by the definition of type. Since $\mathcal{N}_0 \models p$, there exists $e_1, \dots, e_n \in N_0$ such that $\mathcal{N}_0 \models \phi_i(v_1, \dots, v_n, \bar{a})[e_1, \dots, e_n]$, for all $i \in \{1, \dots, m\}$, so we define $c_i^{\mathcal{N}_0} = e_i$. Now, for the second set of sentences, as $\mathcal{M} \models \bigwedge_{i=1}^k \psi_i(\bar{a}, \bar{b})$, then $\mathcal{M} \models \exists w_1 \dots \exists w_l \bigwedge_{i=1}^k \psi_i(\bar{a}, w_1, \dots, w_l)$, which is now an $\mathcal{L}_A$-formula, and as $\mathcal{N}_0 \models \text{Th}_A(\mathcal{M})$, we conclude that:

$$\mathcal{N}_0 \models \exists w_1 \dots \exists w_l \bigwedge_{i=1}^k \psi_i(\bar{a}, w_1, \dots, w_l)$$

So there exists $d_1, \dots, d_l \in N_0$ such that

$$\mathcal{N}_0 \models \bigwedge_{i=1}^k \psi_i(\bar{a}, w_1, \dots, w_l)[d_1, \dots, d_l]$$

By interpreting the constant symbols $b_1, \ldots, b_l$ as $d_1, \ldots, d_l$ we conclude that $\mathcal{N}_0 \models \bigwedge_{i=1}^{k} \psi_i(\bar{a}, \bar{b})$. As for the other constant symbols that were not mentioned, we can interpret them as any element of N_0, as they do not appear in the formulas of Δ, essentially turning $\mathcal{N}_0$ into an $\mathcal{L}_M^*$-structure that models Δ.

By compactness, there exists an $\mathcal{L}_M^*$-structure $\mathcal{N}$ such that $\mathcal{N} \models \Gamma'$ and by ignoring all constant symbols that are not already in $\mathcal{L}$ we get an $\mathcal{L}$-structure that models Γ. $\square$

This proposition allows us to prove the following characterization of complete types which I hinted at earlier:

Corollary 1.7.6. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. Then $p \in S_n^{\mathcal{M}}(A)$ if and only if there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^n$ such that $p = tp^{\mathcal{N}}(\bar{a}/A)$.*

Proof. Let p be an n-type such that $p = \mathrm{tp}^{\mathcal{N}}(\bar{a}/A)$ for some elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^n$. Then, $p \in S_n^{\mathcal{N}}(A) = S_n^{\mathcal{M}}(A)$.

Now, let $p \in S_n^{\mathcal{M}}(A)$. By Theorem 1.7.5, there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^n$ that realizes p, so $p \subseteq \mathrm{tp}^{\mathcal{N}}(\bar{a}/A)$. On the other hand, let $\phi \in \mathrm{tp}^{\mathcal{N}}(\bar{a}/A)$. If $\phi \notin p$ then, as p is a complete type, we would have that $\neg\phi \in p$ and, by the previous inclusion, this would imply that $\neg\phi \in \mathrm{tp}^{\mathcal{N}}(\bar{a}/A)$ which is a contradiction, so $\phi \in p$. $\square$

1.7.2 The Stone space of Complete Types

There is a natural way to give a topology to the set $S_n^{\mathcal{M}}(A)$ of complete n-types over A. Doing so allows us to use tools from topology to study certain kinds of types, one instance of this being isolated types, as we will see later.

Definition 1.7.7. Let $\mathcal{M}$ be a structure and $A \subseteq M$. Let ϕ be an $\mathcal{L}_A$-formula with free variables from $v_1, \ldots, v_n$ and let

$$[\phi] = \{p \in S_n^{\mathcal{M}}(A) : \phi \in p\}$$

The Stone topology in $S_n^{\mathcal{M}}(A)$ is the topology having as a subbase the sets $[\phi]$ for each $\mathcal{L}_A$-formula ϕ with free variables from $v_1, \ldots, v_n$. We call the open sets of the form $[\phi]$ basic open sets.

Note that in this topology, each set of the form $[\phi]$ is also closed, as $[\phi] = S_n^{\mathcal{M}}(A) \setminus [\neg\phi]$.

We will start by justifying the name of the topology we have given to $S_n^{\mathcal{M}}(A)$ i.e. proving that $S_n^{\mathcal{M}}(A)$ is a Stone space, however, we will first need the following lemma:

Lemma 1.7.8. *Let $\mathcal{M}$ be a structure and $A \subseteq M$. Then any n-type p over A can be extended to a complete n-type over A.*

Proof. Let p be any n-type over A and consider the set

$$X = \{q : \text{is an } n\text{-type over } A \text{ and } p \subseteq q\}$$

Let $\Gamma \subseteq X$ be a chain and consider $q_0 = \bigcup_{k \in \Gamma} k$. Let Δ be any finite subset of $q_0 \cup \mathrm{Th}_A(\mathcal{M})$. As the elements of Γ form a linearly ordered set with respect to inclusion, there exists $k \in \Gamma$ such that $\Delta \subseteq k \cup \mathrm{Th}_A(\mathcal{M})$ and because k is a type, Δ is satisfiable. By compactness (implicitly adding constants to the language for each free variable $v_1, \ldots, v_n$ just like we did in the proof of Theorem 1.7.5) we conclude that $q_0 \cup \mathrm{Th}_A(\mathcal{M})$ is satisfiable, and thus q_0 is an upper bound of Γ in X. The conclusion follows from Zorn's lemma. □

Proposition 1.7.9. *$S_n^{\mathcal{M}}(A)$ equipped with the Stone topology is a Stone space, i.e.:*

1. *$S_n^{\mathcal{M}}(A)$ is compact;*
2. *$S_n^{\mathcal{M}}(A)$ is totally disconnected.*

Proof. 1) I will start by showing that $S_n^{\mathcal{M}}(A)$ is Hausdorff. Let $p, q \in S_c^{\mathcal{M}}(A)$ such that $p \neq q$. As both p and q are maximal with respect to inclusion, then $p \not\subseteq q$ and $q \not\subseteq p$, so that we can find ϕ and ψ such that $\phi \in p \setminus q$ and $\psi \in q \setminus p$. This implies, by definition, that $[\phi]$ and $[\psi]$ are open sets that separate p and q.

To prove that $S_n^{\mathcal{M}}(A)$ is compact, by Alexander subbase Theorem, we only need to prove that any cover of basic open sets has a finite subcover.

Let $C = \{[\phi_i] : i \in I\}$ be an open cover and lets assume that C does not have any finite subcover. Consider the set $p = \{\neg\phi_i : i \in I\}$ and let $\Delta \subseteq p \cup \mathrm{Th}_A(\mathcal{M})$ be any finite subset. Let $\Delta = \{\neg\phi_1, \ldots, \neg\phi_k\} \cup \Delta'$, for some $\phi_1, \ldots, \phi_k$ and $\Delta' \subseteq \mathrm{Th}_A(\mathcal{M})$. As $\{\phi_1, \ldots, \phi_n\}$ is not a subcover of C, there exists $q \in S_n^{\mathcal{M}}(A)$ such that $q \notin \bigcup_{i=1}^{k} [\phi_i]$, so that $q \notin [\phi_i]$ and, by definition, $\phi_i \notin q$ for all i. As q is a complete type, this implies that $\neg\phi_i \in q$, for all i. By Theorem 1.7.5, there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ where

q is realized, and so:

$$\begin{aligned} & \mathcal{N} \models \mathsf{Diag}(\mathcal{M}) \cup p \\ \Longrightarrow & \mathcal{N} \models \mathsf{Th}_A(\mathcal{M}) \cup \{\bigwedge_{i=1}^{k} \neg\phi_i\} \\ \Longrightarrow & \mathcal{N} \models \Delta \end{aligned}$$

By compactness, $p \cup \mathsf{Th}_A(\mathcal{M})$ is satisfiable, which, by definition, implies that p is an n-type. By Lemma 1.7.8, there is $q \in S_n^{\mathcal{M}}(A)$ such that $p \subseteq q$. However, as $\neg\phi_i \in q$ for all $i \in I$,

$$q \in S_n^{\mathcal{M}}(A) \setminus \bigcup_{i \in I} [\phi_i]$$

which contradicts the fact that C is a cover.

2) Let C_p be the connected component of $p \in C_p$, and assume that there exists $q \in C_p$ such that $p \neq q$. Then, there exists $\phi \in p$ such that $\neg\phi \in q$. Let $A = C_p \cap [\phi]$ and $B = C_p \cap [\neg\phi]$. Then A, B are disjoint closed sets with $A \cup B = C_p$, contradicting the fact that C_p is connected, hence $C_p = \{p\}$. $\square$

With this topology, we define isolated types as follows.

Definition 1.7.10. We say that a complete n-type $p \in S_n^{\mathcal{M}}(A)$ is *isolated*, if p is an isolated point in the Stone topology, i.e. $\{p\}$ is an open set.

Although this definition relies on the stone topology of $S_n^{\mathcal{M}}(A)$, there exists an alternative characterization of isolated types only using $\mathcal{L}$-formulas, as we will see now.

Proposition 1.7.11. *Let $p \in S_n^{\mathcal{M}}(A)$. The following are equivalent:*

1. *p is an isolated type;*
2. *$\{p\} = [\phi(\bar{v})]$, for some $\mathcal{L}_A$-formula $\phi(\bar{v})$;*
3. *There exists an $\mathcal{L}_A$-formula $\phi(\bar{v})$ such that, for all $\mathcal{L}_A$-formulas $\psi(\bar{v})$,*

$$\psi(\bar{v}) \in p \Leftrightarrow \mathit{Th}_A(\mathcal{M}) \models \forall\bar{v}(\phi(\bar{v}) \rightarrow \psi(\bar{v}))$$

Proof. $(1) \Rightarrow (2)$ Note that, for any two $\mathcal{L}_A$-formulas ϕ and ψ, we have that $[\phi] \cap [\psi] = [\phi \wedge \psi]$ and analogously, $[\phi] \cup [\psi] = [\phi \vee \psi]$.

With this in mind, assume that $\{p\}$ is open. As the basic open sets form a subbase for the Stone topology, the family of all finite intersections of basic open sets forms a basis. So, because $\{p\}$ is open, there are $\mathcal{L}_A$-formulas $\phi_1, \ldots, \phi_k$ such that:

$$p \in \bigcap_{i=1}^{k} [\phi_i] \subseteq \{p\}$$

Which implies that:

$$\{p\} = \bigcap_{i=1}^{k} [\phi_i] = [\phi_1 \wedge \ldots \wedge \phi_k]$$

$(2) \Rightarrow (3)$ Let $p = [\phi(\bar{v})]$ and let $\psi \in p$. Assume, for the sake of contradiction, that $\text{Th}_A(\mathcal{M}) \not\models \forall \bar{v}(\phi(\bar{v}) \to \psi(\bar{v}))$. Then, there exists $\mathcal{N}_0 \models \text{Th}_A(\mathcal{M})$ and $\bar{a} \in N_0^n$ such that $\mathcal{N} \models (\phi \wedge \neg\psi)[\bar{a}]$. This implies that the set $\{\phi, \neg\psi\} \cup \text{Th}_A(\mathcal{M})$ is satisfiable and as such, $\{\phi, \neg\psi\}$ is an n-type. By Theorem 1.7.5, there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^n$ that realizes $\{\phi, \neg\psi\}$. Now, consider the type $\text{tp}^{\mathcal{N}}(\bar{a}/A) \in S_n^{\mathcal{N}}(A) = S_n^{\mathcal{M}}(A)$. Because $\phi \wedge \neg\psi \in \text{tp}^{\mathcal{N}}(\bar{a}/A)$, we have that $\phi \in \text{tp}^{\mathcal{N}}(\bar{a}/A)$, and because $[\phi] = \{p\}$, this implies that $\text{tp}^{\mathcal{N}}(\bar{a}/A) = p$. However, $\phi \wedge \neg\psi \in \text{tp}^{\mathcal{N}}(\bar{a}/A)$ also implies that $\neg\psi \in \text{tp}^{\mathcal{N}}(\bar{a}/A) = p$, which contradicts $\psi \in p$.

On the other hand, assume that $\text{Th}_A(\mathcal{M}) \models \forall \bar{v}(\phi(\bar{v}) \to \psi(\bar{v}))$, for some ψ. If $\psi \notin p$, then $\neg\psi \in p$, and as we say, this would imply that $\text{Th}_A(\mathcal{M}) \models \forall \bar{v}(\phi(\bar{v}) \to \neg\psi(\bar{v}))$. By Theorem 1.7.5, there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^n$ that realizes p, and in particular $\mathcal{N} \models \phi[\bar{a}]$. Because $\mathcal{M} \preceq \mathcal{N}$, we have that $\mathcal{N} \models \text{Th}_A(\mathcal{M})$ and as such, $\mathcal{N} \models \psi[\bar{a}]$ and $\mathcal{N} \models \neg\psi[\bar{a}]$, which is a contradiction.

$(3) \Rightarrow (1)$ Assuming that (3) holds, we will show that $\{p\} = [\phi]$. As $\text{Th}_A(\mathcal{M}) \models \forall \bar{v}(\phi(\bar{v}) \to \phi(\bar{v}))$, we have that $\phi \in p$ and therefore, $\{p\} \subseteq [\phi]$.

Now, let $q \in [\phi]$. We will prove that $q = p$.

If $\psi \in p$, then $\text{Th}_A(\mathcal{M}) \models \forall \bar{v}(\phi(\bar{v}) \to \psi(\bar{v}))$. For the sake of contradiction, assume that $\psi \notin q$, i.e. $\neg\psi \in q$. Because $\phi \in q$, then we would have that $\phi \wedge \neg\psi \in q$. This is equivalent to $\neg(\phi \to \psi) \in q$ which would be a contradiction because q is realized in some elementary extension of $\mathcal{M}$ that models $\text{Th}_A(\mathcal{M})$, so we conclude that $\psi \in q$.

On the hand let $\psi \in q$ and assume that $\psi \notin p$. Then $\neg\psi \in p$ and by our hypothesis, $\text{Th}_A(\mathcal{M}) \models \forall \bar{v}(\phi(\bar{v}) \to \neg\psi(\bar{v}))$. Again, by Theorem 1.7.5, there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ and $\bar{a} \in N^n$ that realizes q, and in particular $\mathcal{N} \models \phi[\bar{a}]$ and $\mathcal{N} \models \psi[\bar{a}]$. However, $\mathcal{N} \models \forall \bar{v}(\phi(\bar{v}) \to \neg\psi(\bar{v}))$, meaning that $\mathcal{N} \models \neg\psi[\bar{a}]$, which is a contradiction.

So we conclude that $\psi \in p$. $\square$

If p is an isolated type and $\{p\} = [\phi]$, we say that ϕ isolates p.

1.7.3 The Omitting Types Theorem

As I said at the beginning of this section, the other main theorem we will discuss is the omitting types theorem. However, contrary to the realizing types theorem, we will not use the omitting types theorem in this work. Nonetheless, for the sake of completion, I will briefly discuss it.

Our study of types until now was all done using the realizing types theorem. A natural question to ask is: Given any type p, is it possible to find a structure that omits this type? The answer is negative, as we will see. However, there are some types that we can always omit in some structure as we will see later with the omitting types theorem.

Start by noting that, by the definition of type, all the types we were working with are "bound" to a choice of a structure $\mathcal{M}$, so we will start with the following generalization:

Definition 1.7.12. Let T be an $\mathcal{L}$-theory. An n-type of T is a set of $\mathcal{L}$-formulas p with free variables from $v_1, \ldots, v_n$ such that $p \cup T$ is satisfiable.

Given an $\mathcal{L}$-structure $\mathcal{M}$, and $A \subseteq M$, then by taking the language to be $\mathcal{L}_A$ and $T = \mathrm{Th}_A(\mathcal{M})$, we get the previous definition of type we have been using.

The definition of complete and incomplete types still makes sense in this context so we will keep those as they are. We denote the set of all complete n-types of T as $S_n(T)$. Note that, using this notation, if T is complete and $\mathcal{M} \models T$, then $S_n(T) = S_n^{\mathcal{M}}(\emptyset)$.

We can still give a topology to $S_n(T)$ by defining the same subbase we used in $S_n^{\mathcal{M}}(A)$, and in fact, this is still a Stone space. We define the isolated types in $S_n(T)$ the same way we did in $S_n^{\mathcal{M}}(A)$. In particular, Proposition 1.7.11 still holds in $S_n(T)$, with the minor adjustment of substituting $\mathrm{Th}_A(\mathcal{M})$ for T in (3).

Now, we would like to generalize the notion of isolated types to possibly incomplete ones. However, the space of all n-types is not equipped with a topology, so we need an alternative definition. Point (3) of Proposition 1.7.11 motivates the following:

Definition 1.7.13. Let T be an $\mathcal{L}$-theory and p a (possibly incomplete) n-type of T. We say that p is *isolated* if there exists a $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n)$ such that $T \cup \{\phi(\bar{v})\}$ is satisfiable and, for all $\psi(\bar{v}) \in p$,

$$T \models \forall \bar{v}(\phi(\bar{v}) \rightarrow \psi(\bar{v}))$$

If this is the case, we say that ϕ isolated p.

Note that, if p is a complete n-type, by the proof that $(2) \Rightarrow (3)$ in Proposition 1.7.11, p being isolated in this sense implies that p is isolated as defined in definition 1.7.10.

With this in mind, we have the following result:

Proposition 1.7.14. *Let T be an $\mathcal{L}$-theory and p be any n-type of T. If $\phi(\bar{v})$ isolates p, then p is realized in any model of $T \cup \{\exists \bar{v}\phi(\bar{v})\}$. In particular, if T is complete, then every model of T realizes every isolated type.*

Proof. Let $\mathcal{M} \models T \cup \{\exists \bar{v}\phi(\bar{v})\}$. Then, there exists $\bar{a} \in M^n$ such that $\mathcal{M} \models \phi[\bar{a}]$. Let $\psi(\bar{v}) \in p$. Then $\mathcal{M} \models \forall \bar{v}(\phi(\bar{v}) \rightarrow \psi(\bar{v}))$ and as such, $\mathcal{M} \models \psi[\bar{a}]$, meaning that $\bar{a}$ realizes p.

Now assume that T is complete, that $\mathcal{M} \models T$ and that p is an n-type isolated by ϕ. By Definition 1.7.13, there exists $\mathcal{N}$ that models $T \cup \{\phi\}$, meaning that $\mathcal{N} \models T$ and $\mathcal{N} \models \exists \bar{v}\phi(\bar{v})$. Because T is complete, $\mathcal{N} \equiv \mathcal{M}$, meaning that $\mathcal{M} \models \exists \bar{v}\phi(\bar{v})$. The conclusion follows from the first part of the proof. □

So as we just saw, in a complete theory, there is no way to avoid realizing isolated types. However, the same is not true when we look at non-isolated types, and this is precisely the statement of the omitting types theorem:

Theorem 1.7.15 (Omitting Types Theorem)**.** *Let $\mathcal{L}$ be a countable language, T a $\mathcal{L}$-theory and p a (possibly incomplete) non-isolated type. Then there exists a countable $\mathcal{N} \models T$ that omits p.*

Proof. As we will not use this result, I will not present a proof. You can refer to Theorem 4.2.3 of [5] for the proof. □

1.8. Saturated, Homogeneous and Universal Structures

For us, in this work, one of the main uses of types will be the fact that they allow us to define and work with a special kind of structures known as saturated structures. In fact, when studying definable groups and semigroups in o-minimal structures, we will often assume that aside from o-minimal, our structure of interest is also saturated, as the properties provided by saturation are essential to prove most results more easily.

Intuitively, a saturated structure is characterized by being very ”large” in the sense that, if an element can exist within the structure without creating any contradictions, then it is already in the structure, i.e., any type is realized.

In this subsection we will only introduce key properties that we will use, so keep in mind that the general theory of saturated, homogeneous and universal structures is much deeper than what we will discuss here.

We will start by formally defining these three classes of structures.

Definition 1.8.1 (Saturated Structure)**.** Let $\mathcal{M}$ be a structure and κ an infinite cardinal. We say that $\mathcal{M}$ is *κ-saturated* if for all $A \subseteq M$ with $|A| < \kappa$, all the types from $S_n^{\mathcal{M}}(A)$ are realized in $\mathcal{M}$, for any $n \geq 1$.

We say that $\mathcal{M}$ is *saturated* if it is $|M|$-saturated.

Definition 1.8.2 (Homogeneous Structure)**.** Let κ be an infinite cardinal. We say that a structure $\mathcal{M}$ is *κ-homogeneous* if given:

- $A \subseteq M$ with $|A| < \kappa$;
- a partial elementary map $f : A \to M$;
- $b \in M$.

There exists a partial elementary map $f^* : A \cup \{b\} \to M$ that extends f, i.e. $f \subseteq f^*$.

We say that $\mathcal{M}$ is *homogeneous* if it is $|M|$-homogeneous.

Definition 1.8.3 (Universal Structure)**.** Let κ be an infinite cardinal and T be a theory. We say that a model $\mathcal{M}$ of T is *κ-universal*, if every model $\mathcal{N}$ of T with $|N| < \kappa$ embeds into $\mathcal{M}$. We say that $\mathcal{M}$ is *universal* if it is $|M|^+$-universal.

The reason why I introduced these three classes of structures at the same time is that they are not as unrelated as they first may seem. Before explaining what I mean by this, we need the following technical lemma which gives an alternative characterization of saturation.

Lemma 1.8.4. *Let $\mathcal{M}$ be a structure and κ be an infinite cardinal. The following are equivalent:*

1. *$\mathcal{M}$ is κ-saturated;*

2. *If $A \subseteq M$ with $|A| < \kappa$, then every type in $S_1^{\mathcal{M}}(A)$ is realized in $\mathcal{M}$.*

Proof. $(1) \Rightarrow (2)$ by definition of saturation.

On the other hand, assume that (2) is true.

Let $A \subseteq M$ with $|A| < \kappa$. We prove that every type in $S_n^{\mathcal{M}}(A)$ is realized in $\mathcal{M}$ using induction over n.

If $n = 1$, then this is true by (2).

Assume now that, for any $B \subseteq M$ with $|B| < \kappa$, every type in $S_{n-1}^{\mathcal{M}}(B)$ is realized in $\mathcal{M}$ and let $p \in S_n^{\mathcal{M}}(A)$. Consider the type $q \in S_{n-1}^{\mathcal{M}}(A)$ given by

$$q = \{\phi(v_1, \ldots, v_{n-1}) : \phi \in p\}$$

By the induction hypothesis, there exists $\bar{a} \in M^{n-1}$ that realizes q. Now, consider the type $t \in S_1^{\mathcal{M}}(A \cup \bar{a})$ given by

$$t = \{\, \phi(\bar{a}, v_n) : \phi(v_1, \ldots, v_n) \in p \,\}$$

where $\phi(\bar{a}, v)$ is the formula obtained from ϕ by substituting every free occurrence of $v_1, \ldots, v_{n-1}$ by $a_1, \ldots, a_{n-1}$ respectively. By (2), there exists $b \in M$ that realizes t, and as such, $(\bar{a}, b)$ realizes p. $\square$

With this lemma, we can prove the following theorem which explains how saturated, homogeneous and universal structures relate to each other.

Theorem 1.8.5. *Let κ be an infinite cardinal, $\mathcal{L}$ be a countable language, T be an $\mathcal{L}$-theory and $\mathcal{M}$ be a structure with $\mathcal{M} \models T$. The following are equivalent:*

1. *$\mathcal{M}$ is κ-saturated;*

2. *$\mathcal{M}$ is κ-homogeneous and κ^+-universal;*

Additionally, if $\kappa \geq \aleph_1$, (1) and (2) are also equivalent to:

3. *$\mathcal{M}$ is κ-homogeneous and κ-universal.*

Proof. Start by noting that $(2) \Rightarrow (3)$ trivially.

$(1) \Rightarrow (2)$:

Assume that $\mathcal{M}$ is κ-saturated for some infinite cardinal κ.

Claim: $\mathcal{M}$ is κ-homogeneous.

Let $A \subseteq M$ with $|A| < \kappa$, $f : A \to M$ be a partial elementary map and $b \in M \setminus A$. The goal is to show that there exists a partial elementary map $f^* : A \cup \{ b \} \to M$ such that $f \subseteq f^*$.

For $n \geq 0$, for a formula $\phi(v, w_1, \dots w_n)$ and $\bar{a} \in A^n$ such that $\mathcal{M} \models \phi[b, a_1, \dots, a_n]$, consider the formula $\phi(v, f(a_1), \dots, f(a_n))$ obtained from ϕ by substituting all free occurrences of each variable w_i by the constant $f(a_i)$ and let Γ be the set consisting of the formulas obtained this way. In particular, Γ is a set of $\mathcal{L}_{f(A)}$-formulas with one free variable, which is a 1-type over $f(A)$ as we will see now.

Start by noting that Γ is closed under finite conjunctions, which implies that, to show that Γ is satisfiable by compactness, it is enough to show that each individual formula of Γ is satisfiable. Let $\phi(v, f(\bar{a})) \in \Gamma$ which, by definition of Γ means that $\mathcal{M} \models \phi[b, \bar{a}]$. In particular, $\mathcal{M} \models \exists v \phi(v, \bar{w})[a]$ and as f is a partial elementary embedding, $\mathcal{M} \models \exists v \phi(v, \bar{w})[f(\bar{a})]$, i.e. $\mathcal{M} \models \phi(v, f(\bar{a}))[c]$ for some $c \in M$. Hence, by compactness, Γ is a 1-type over $f(A)$.

By Lemma 1.7.8, there exists a complete 1-type p over $f(A)$ such that $\Gamma \subseteq p$. As $|f(A)| < \kappa$, by saturation, there exists $c \in M$ that realizes p and therefore it also realizes Γ. Let $f^* : A \cup \{ b \} \to M$ be the partial map defined as $a \mapsto f(a)$ for $a \in A$ and $b \mapsto c$. Then f^* is a partial elementary map extending f, concluding the proof that $\mathcal{M}$ is κ-homogeneous.

Claim: $\mathcal{M}$ is κ^+-universal.

Let $\mathcal{N} \models T$ with $|N| \leq \kappa$. We will build an elementary embedding $\mathcal{N} \hookrightarrow \mathcal{M}$. Let $(x_\alpha : \alpha < |N|)$ be an enumeration of $\mathcal{N}$ and for each $\alpha < |N|$ let A_α be the set $\{x_\beta : \beta < \alpha\}$. The idea is to build a chain of partial elementary maps $f_0 \subseteq f_1 \subseteq \dots \subseteq f_\alpha \subseteq \dots$ for each $\alpha < |N|$ with $f_\alpha : A_\alpha \to M$ and then take f to be $\bigcup_{\alpha < |N|} f_\alpha$.

Let $f_0 = \emptyset$. For each $\alpha < |N|$, we define f_α recursively as follows:

If α is a limit ordinal, then $f_\alpha = \bigcup_{\beta < \alpha} f_\beta$. On the other hand, let α is a successor ordinal, i.e. $\alpha = \beta + 1$. For $n \geq 0$, for a formula $\phi(v, w_1, \dots, w_n)$ and $\bar{a} \in A_\beta^n$ such that $\mathcal{M} \models \phi[x_\alpha, \bar{a}]$, consider the formula $\phi(v, f_\beta(\bar{a}))$ which is obtained from ϕ by substituting all free occurrences of each variable w_i by $f_\beta(a_i)$, and let Γ be the set of $\mathcal{L}_{f_\beta(A_\beta)}$-formulas obtained this way. Note that $|f_\beta(A_\beta)| \leq |A_\beta| \leq |A| \leq \kappa$ and thus, by the same argument we did before when proving that $\mathcal{M}$ is κ-homogeneous, we conclude that Γ is a 1-type over $f_\beta(A_\beta)$. By Lemma 1.7.8, we can extend it to a complete 1-type, and by saturation there exists some $c \in M$ that realizes it and, in particular, realizes Γ. Define $f_\alpha = f_\beta \cup \{(x_\alpha, c)\}$. Then f_α is by definition a partial elementary map.

As I said before, by taking $f = \bigcup_{\alpha<|N|} f_\alpha$ we get an elementary embedding from $\mathcal{N}$ into $\mathcal{M}$, thus proving that $\mathcal{M}$ is κ^+-universal.

$(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$ if κ is uncountable:

Let $A \subseteq M$ with $|A| < \kappa$. We will show that every type in $S_1^{\mathcal{M}}(A)$ is realized, which, by Lemma 1.8.4, implies that $\mathcal{M}$ is κ-saturated.

Let $p \in S_1^{\mathcal{M}}(A)$. By definition, p being a 1-type of $\mathcal{M}$ over A means that $p \cup \text{Th}_A(\mathcal{M})$ is satisfiable. Note that if $\kappa = \aleph_0$ then $|\mathcal{L}_A| = \aleph_0$, and if $\kappa \geq \aleph_1$ then $|\mathcal{L}_A| < \kappa$. Thus, by Theorem 1.3.10, there exists $\mathcal{N} \models \text{Th}_A(\mathcal{M})$ where p is realized such that if $\kappa = \aleph_0$, then $|N| = \aleph_0$ and if $\kappa \geq \aleph_1$, then $|N| < \kappa$. If $\kappa = \aleph_0$, then hypothesis (2) implies the existence of an elementary embedding $f : \mathcal{N} \to \mathcal{M}$. On the other hand, if $\kappa \geq \aleph_1$, to grantee the existence of an elementary embedding $f : \mathcal{N} \to \mathcal{M}$, it is enough to assume only that $\mathcal{M}$ is κ-universal, i.e. hypothesis (3). Also, note that the elementary embedding f is an elementary map with respect to the original language $\mathcal{L}$ and not $\mathcal{L}_A$.

Let $x \in N$ realize p. If $x \in A$ then, as $A \subseteq M$, p is realized in $\mathcal{M}$ and we are done. Assume, on the other hand that, $x \notin A$. Note that the restriction $f|_A : A \to M$ is a partial elementary map and thus, injective. Consider the partial elementary map $f|_A^{-1} : f(A) \to M$. By κ-homogeneity, there exists a partial elementary map $f^* : f(A) \cup \{ f(x) \} \to M$ that extends $f|_A^{-1}$.

Claim: $b := f^*(f(x))$ realizes p

Let $\phi(v) \in p$ be an $\mathcal{L}_A$-formula. As x realizes p, we have that $\mathcal{N} \models \phi[x]$. Let $a_1, \ldots, a_k \in A$ be the constants from A that appear in $\phi(v)$, and consider the formula $\phi_0(v, w_1, \ldots, w_k)$ obtained from ϕ by substituting each occurrence of a_i by a new free variable w_i, for each $i = 1, \ldots, k$. Then, $\mathcal{N} \models \phi_0[x, \bar{a}]$ and as f is an elementary embedding, $\mathcal{M} \models \phi_0[f(x), f(\bar{a})]$. Again, as f^* is a partial elementary embedding, $\mathcal{M} \models \phi_0[f^*(f(x)), f^*(f(\bar{a}))]$. Now, note that by definition $f^*(f(x)) = b$ and $f^*(f(\bar{a})) = \bar{a}$, i.e. $\mathcal{M} \models \phi_0[b, \bar{a}]$ and, in particular, $\mathcal{M} \models \phi[b]$. As $\phi(v) \in p$ is arbitrary, we conclude that b realizes p in $\mathcal{M}$. $\square$

As a consequence of this theorem, we have the following important characterization of saturated structures.

Corollary 1.8.6. *A structure $\mathcal{M}$ is saturated if and only if it is homogeneous and universal.*

It is common in the literature to see statements like "Let $\mathcal{M}$ be a sufficiently saturated structure such that ..." or "[some statement] can be proved by passing to a sufficiently saturated elementary extension of $\mathcal{M}$". A structure being sufficiently saturated means that it is κ-saturated for some κ larger than any cardinality appearing in the rest of the argument or proof. For example, if a proof involves only working with a countable number of countably infinite sets, we might want the structure we are working with to be at least $\aleph_1$-saturated or, analogously, if we need to work with a collection of λ sets each with cardinality of at most λ, it is useful to consider that the structure is at least λ^+-saturated. This raises the question of whether κ-saturated structures exist for arbitrarily large κ. The existence of arbitrarily large saturated structures is very technical and depends heavily on the set-theoretic assumptions we make. For example, there are some results about the existence of such saturated structures that assume that the continuum hypothesis is true, and others that assume, for example, assume the existence of strongly inaccessible cardinals. As such, we will not delve much into this issue and will instead assume that there are arbitrarily large saturated structure. For some results about this issue see the section "Existence of Saturated Models" in [5].

Note also that if we are working inside some structure $\mathcal{M}$, we can at any time consider a sufficiently saturated structure $\mathcal{N}$ with cardinality strictly larger then that of $\mathcal{M}$. As saturated structures also also universal, and as $|M| < |N|$, there exists an elementary embedding $\mathcal{M} \hookrightarrow \mathcal{N}$, meaning that without loss of generality we may assume that $\mathcal{M} \preceq \mathcal{N}$.

To wrap up this section, we will prove some properties of saturated structures that will be crucial in some of the proofs we present later.

Lemma 1.8.7. *Let $\mathcal{M}$ be sufficiently saturated, $D \subseteq M^n$ be definable, and Σ be a collection of definable subsets of M^n. If $D \subseteq \bigcup_{U \in \Sigma} U$, then there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $D \subseteq \bigcup_{U \in \Sigma_0} U$.*

Proof. We will use the compactness theorem indirectly by using the fact that the space of types is compact. Let A be a set such that all elements of Σ and D are A-definable. For each $U \in \Sigma$, denote also by U the $\mathcal{L}_A$-formula that defines U, i.e. $U = \{x \in M^n : \mathcal{M} \models U[x]\}$. Analogously, let D denote the $\mathcal{L}_A$-formula that defines D.

I will start by showing that $[D] \subseteq \bigcup_{U\in\Sigma}[U]$ (Recall definition 1.7.7). Let $p \in [D]$, i.e. $D \in p$ and let $a \in M^n$ realize p, which exists by saturation of $\mathcal{M}$. As $\mathcal{M} \models D[a]$, we have that $a \in D$ and as the sets of Σ cover D, there exists $U \in \Sigma$ such that $a \in U$. Then $\mathcal{M} \models U[a]$, and this $U \in \mathrm{tp}^{\mathcal{M}}(a/A) = p$, i.e. $p \in [U]$.

As we have seen the set $[D]$ is closed and $S_n^{\mathcal{M}}(A)$ is compact, so there exists $U_1, \ldots, U_k \in \Sigma$ such that $[D] \subseteq [U_1] \cup \ldots \cup [U_k]$. The only thing left to show is that $D \subseteq U_1 \cup \ldots \cup D_k$.

Let $a \in D$. Then $D \in \mathrm{tp}^{\mathcal{M}}(a/A)$ and so $\mathrm{tp}^{\mathcal{M}}(a/A) \in [D] \subseteq [U_1] \cup \ldots \cup [U_k]$. This implies that $\mathrm{tp}^{\mathcal{M}}(a/A) \in [U_i]$ for some i. By definition, $U_i \in \mathrm{tp}^{\mathcal{M}}(a/A)$ which means that $\mathcal{M} \models U_i[a]$, i.e. $a \in U_i$. $\square$

The next two properties, although true in any saturated structure, only require that the structure is homogeneous, and thus we state them with this weaker hypothesis.

Proposition 1.8.8. *Let $\mathcal{M}$ be homogeneous, $A \subseteq M$ with $|A| < |M|$ and $f : A \to M$ be a partial elementary map. Then there exists an automorphism $\pi : M \to M$ such that $f \subseteq \pi$.*

Proof. Let $|M| = \kappa$ and let $(x_\alpha : \alpha < \kappa)$ be an enumeration of M. We will start by building a chain of partial elementary maps $f \subseteq f_1 \subseteq \ldots \subseteq f_\alpha \subseteq \ldots$ such that, for each $\alpha < \kappa$, x_α is in the domain and image of $f_{\alpha+1}$ and $|f_\alpha| < \kappa$.

Start by setting $f_0 = f$, which by hypothesis has $|f_0| = |A| < \kappa$.

If $\alpha < \kappa$ is a limit ordinal, set $f_\alpha = \bigcup_\beta f_\beta$, and in particular, as $|f_\beta| < \kappa$ for all $\beta < \alpha$, we have that $|f_\alpha| < |\alpha| \cdot \kappa = \kappa$.

On the other hand, assume that f_α is already defined with $|f_\alpha| < \kappa$. As $\mathcal{M}$ is homogeneous there exists $b \in M$ such that $g_\alpha = f_\alpha \cup \{(x_\alpha, b)\}$ is a partial elementary map. As partial elementary maps are injective, consider the partial elementary map given by g_α^{-1}. Again, by homogeneity, there exists $c \in M$ such that $g_\alpha^{-1} \cup \{(x_\alpha, c)\}$ is a partial elementary map. We can then define $f_{\alpha+1}$ as $f_\alpha \cup \{(x_\alpha, b), (c, x_\alpha)\}$ which is a partial elementary map with $|f_{\alpha+1}| \leq |f_\alpha| + 2 < \kappa$.

To conclude the proof, we let $\pi = \bigcup_{\alpha<\kappa} f_\alpha$, which is an automorphism extending f. $\square$

This proposition allows us to conclude the following about points with the same complete type over a small A in an homogeneous structure:

Corollary 1.8.9. *Let $\mathcal{M}$ be a homogeneous structure, $A \subseteq M$ with $|A| < |M|$, and let $\bar{a}, \bar{b} \in M^n$. Then tp$^{\mathcal{M}}(\bar{a}/A) =$ tp$^{\mathcal{M}}(\bar{b}/A)$ if and only if there exists an automorphism $\pi : M \to M$ with $\pi(\bar{a}) = \bar{b}$ that fixes A pointwise.*

Proof. Start by expanding the language to $\mathcal{L}_A$, including the elements of A as constants, and note that any $\mathcal{L}_A$-automorphism of $\mathcal{M}$ fixes by definition the elements of A pointwise. As such, without loss of generality, we may assume that $A = \emptyset$.

Assume that tp$^{\mathcal{M}}(\bar{a}) =$ tp$^{\mathcal{M}}(\bar{b})$ and consider the partial map f that simply maps $\bar{a} \mapsto \bar{b}$. Note that for any $\mathcal{L}$-formula ϕ,

$$\begin{aligned}
\mathcal{M} \models \phi[\bar{a}] &\Leftrightarrow \phi \in \text{tp}^{\mathcal{M}}(\bar{a}) \\
&\Leftrightarrow \phi \in \text{tp}^{\mathcal{M}}(\bar{b}) \\
&\Leftrightarrow \mathcal{M} \models \phi[\bar{b}] \\
&\Leftrightarrow \mathcal{M} \models \phi[f(\bar{a})]
\end{aligned}$$

By Proposition1.8.8, there exists an automorphism that sends $\bar{a}$ to $\bar{b}$.

On the other hand, as any automorphism is an elementary map, if the image of $\bar{a}$ under some automorphism is $\bar{b}$, then tp$^{\mathcal{M}}(a) =$ tp$^{\mathcal{M}}(\bar{b})$. □

For the sake of completeness, I will also state the following theorem. For the proof see Theorem 4.3.20 of [5].

Theorem 1.8.10. *Any two saturated models of T with the same cardinality are isomorphic.*

1.9. Definable choice and Definable Skolem Functions

The last model-theoretic tool I want to introduce in this chapter is that of elimination of imaginaries, definable choice and definable Skolem functions. These concepts are somewhat similar in the sense that they allows us to ”definably chose” an element from a family of definable sets. We will now see more precisely how they are defined rigorously and how they relate to each other.

Definition 1.9.1. Let $\mathcal{M}$ be a structure. A set I is said to be *interpretable* in $\mathcal{M}$ if $I = S/E$, for some definable set $S \subseteq M^n$ and some definable equivalence relation E on S.

We say that a function f is interpretable if its graph $\Gamma(f)$ is interpretable, and a relation R is interpretable if it is interpretable as a set.

Although interpretable sets are not generally subsets of M^n for some n, in certain cases, we can identify its elements with elements from a definable set. The following definition will make this more precise.

Definition 1.9.2. Let $\mathcal{M}$ be a structure. We say that $\mathcal{M}$ has *elimination of imaginaries* (EI) if, for every definable equivalence relation E on M^n, there exists a definable set $D \subseteq M^k$, for some k and a definable function $f : M^n \to D$ such that:

$$xEy \Leftrightarrow f(x) = f(y)$$

Note that if $\mathcal{M}$ has EI, $S \subseteq M^n$ is definable and E is a definable equivalence relation on S, we can extend E to a definable equivalence relation on $E' \subseteq M^n$ as:

$$E' = E \cup \{(x, x) : x \in M^n\}$$

Elimination of imaginaries grantees the existence of a definable map $f' : M^n \to Y$ for some definable $Y \subseteq M^k$. By restricting the domain to S we get a function $f = f'|_S : S \to Y$ such that

$$xEy \Leftrightarrow f(x) = f(y)$$

which in turn induces a bijection g between S/E and $\mathrm{Im}(f)$

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[dr, "f"'] & S/E \arrow[d, "g"] \\ & \mathrm{Im}(f) \end{tikzcd}$$

So we can identify S/E with $\mathrm{Im}(f)$ which is definable, and by doing this identification we sometimes treat the set S/E as being definable when in reality we are referring to $\mathrm{Im}(f)$.

Before proceeding, recall that a *definable family* of sets is a family of the form $\{\phi(\mathcal{M}, a) : a \in \psi(\mathcal{M})\}$ for some formulas $\phi(x, a)$ and $\psi(x)$, i.e. a definable family is a set of uniformly definable sets parametrized by a definable set.

Definition 1.9.3. Let $S \subseteq M^n, X \subseteq M^k$ be definable, and for each $x \in X$ let T_x be uniformly definable subsets of S.

We say that the definable family $T = \{T_x : x \in X\}$ has *definable Skolem functions* if there exists a definable map $f : X \to S$ such that for all $t \in T$, $f(t) \in X_t$.

Additionally, we say the definable family $T = \{T_x : x \in X\}$ has *definable choice* if there exists a definable function $f : X \to S$ such that for all $x \in X$ we have $f(x) \in T_x$, and for all $x, y \in X$, if $T_x = T_y$ then $f(x) = f(y)$.

A definable set S has definable choice [definable Skolem functions] if any definable family of definable subsets of S has definable choice [definable Skolem functions].

Furthermore, we say that $\mathcal{M}$ has definable choice [definable Skolem functions] if M^n has definable choice [definable Skolem functions] for all $n \in \mathbb{N}$.

Note that by definition, if $\mathcal{M}$ has definable choice, then it has definable Skolem functions.

Definable choice is a stronger assumption than elimination of imaginaries. In particular, we have the following.

Proposition 1.9.4. *If $\mathcal{M}$ has definable choice, then $\mathcal{M}$ has elimination of imaginaries.*

Proof. Let $n \in \mathbb{N}$ and E be an equivalence relation on M^n. Consider the definable family

$$T = \{[x]_E : x \in M^n\}$$

By definable choice, there exists a definable function $f : M^n \to M^n$ such that $f(x) \in [x]_E$ for all $x \in M^n$ and $[x]_E = [y]_E \Rightarrow f(x) = f(y)$. Such function satisfies

$$xEy \Leftrightarrow f(x) = f(y) \qquad \square$$

The main benefit that definable choice has over elimination of imaginaries, other than not being limited to quotients of definable sets over definable equivalence relations, is the following: Let S/E be interpretable in a structure $\mathcal{M}$ with definable choice. We already know from EI that there is a bijection between S/E and a definable set Y. Definable choice adds to this by saying that we may choose the definable set Y such that it consists exactly of one representative from each equivalence class of E.

2. O-minimality

We now introduce the notion of o-minimality, which plays a central role in this work. Intuitively, a structure is o-minimal if the definable sets are 'simple' and do not exhibit any pathological behaviour. As such, o-minimality provides a framework to study 'tame' geometric and topological behaviour. In this Chapter we use the model theoretic tools we developed in Chapter 1 to define and derive some key properties of o-minimal structures.

Before starting, I would like to fix the following:

> For the rest of this work, unless stated otherwise, "definable" will always mean definable with parameters.

We will start by introducing some conventions, notations and basic facts about linearly ordered structures which we will use thoroughly when studying o-minimality.

A linearly ordered structure is a structure $\mathcal{M} = (M, <, \ldots)$ where $<$ is a linear order on M. Two important classes of subsets we usually work with when studying linearly ordered structures are convex subsets and intervals:

Definition 2.0.1. Let $\mathcal{M} = (M, <, \ldots)$ be a linearly ordered structure. We say that a subset $A \subseteq M$ is *convex*, if for any $x, y \in A$ and $z \in M$, then $x < z < y$ implies that $z \in A$. On the other hand, an *interval* is any of the following sets:

$$(a, b) := \{x : a < x < b\}$$
$$[a, b] := \{x : a \leq x \leq b\}$$
$$[a, b) := \{x : a \leq x < b\}$$
$$(a, b] := \{x : a < x \leq b\}$$

with $a, b \in M \cup \{-\infty, \infty\}$, where $\{-\infty, \infty\}$ are new elements that act as the endpoints of the other in M. The intervals of the form (a, b) are called open intervals, and the ones of the form $[a, b]$ are called closed intervals.

Note that all intervals are convex, but the converse is not true. Consider for example the linear order $(\mathbb{Q}, <)$, and the subset $\{x \in \mathbb{Q} : x^2 < 2\} = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$. This is convex but it is not an interval in $(\mathbb{Q}, <)$ since we can not write it as (a, b) for some $a, b \in \mathbb{Q}$.

The order topology in a linearly ordered structure $\mathcal{M} = (M, <, \ldots)$ is the topology having all open intervals as a basis, and we consider the product topology on M^n. In particular, note that for any $n \geq 1$, M^n has a definable basis for this topology.

For example, in $\mathcal{R} = (\mathbb{R}, <)$, the other topology is the usual Euclidean topology.

One advantage of working with linear orders is the fact that algebraic closure is the same as definable closure.

Proposition 2.0.2. *Let $\mathcal{M} = (M, <, \ldots)$ be a linearly ordered structure and $A \subseteq M$. Then dcl$(A) =$ acl(A).*

Proof. We have already seen that dcl$(A) \subseteq$ acl(A).

On the other hand, let $x \in$ acl(A). By definition, there exists $\bar{a} \in A$ and a formula $\psi(v, \bar{w})$ such that $\mathcal{M} \models \psi[x, \bar{a}]$ and $\psi(\mathcal{M}, \bar{a})$ is finite. Let $\psi(\mathcal{M}, \bar{a}) = \{x_1, \ldots, x_k\}$ and, without loss of generality, assume that $x_1 < \ldots < x_k$

The idea is that we can use the order $<$ to chose any x_i we want from $\psi(\mathcal{M}, \bar{a})$. For this, consider the formulas $\phi_1, \ldots, \phi_k$ defined inductively as:

$$\phi_1(v, \bar{w}) = \psi(v, \bar{w}) \wedge \forall z(\psi(z, \bar{w}) \rightarrow v \leq z)$$

$$\phi_{n+1}(v, \bar{w}) = \psi(v, \bar{w}) \wedge \forall z \left(\left(\psi(z, \bar{w}) \wedge \bigwedge_{i=1}^{n} \neg\phi_i(z, \bar{w}) \right) \rightarrow v \leq z \right)$$

By definition, $y = x_i$ if and only if $\mathcal{M} \models \phi_i[y, \bar{a}]$, i.e. each x_i is A-definable and in particular so is x. $\square$

2.1. O-minimal Structures

O-minimal structures are a special class of linearly ordered structures where the complexity of definable sets is strictly controlled, as we will see now.

Definition 2.1.1. Consider a structure $\mathcal{M} = (M, <, \ldots)$, where $<$ is a dense linear order on M. We say that $\mathcal{M}$ is *o-minimal* if every definable subset of M is a finite union of singletons and open intervals (with endpoints in $M \cup \{-\infty, \infty\}$).

Roughly speaking, a structure is o-minimal if every definable set can be defined by a quantifier-free formula only using the relational symbol $<$ and $=$.

Example 2.1.2. *Some examples of o-minimal structures include:*

- *Any dense linear order without endpoints, as DLO has quantifier elimination;*
- $(\mathbb{Q}, <, +)$, *by quantifier elimination of the theory of divisible ordered abelian groups;*
- $(\mathbb{R}, <, +, -, \cdot, 0, 1)$, *by quantifier elimination (See Corollary 3.3.23 in [5]);*
- $(\mathbb{R}, <, \exp)$, *by A. Wilkie's work [14].*

An example of a structure that is not o-minimal would be $(\mathbb{R}, \sin)$, *as the set* $\{x \in \mathbb{R} : \sin x = 0\}$ *is definable but not a finite union of points and intervals.*

2.1.1 The Monotonicity Theorem

The following result is of paramount importance to the theory of o-minimality. This theorem determines the behavior of definable one-variable functions in o-minimal structures.

Theorem 2.1.3 (The Monotonicity Theorem)**.** *Let $\mathcal{M}$ be an o-minimal structure, $a, b \in M$ and $f : (a, b) \to M$ a definable function (where the endpoints could be $\pm\infty$). Then there are points $a = a_0 < \ldots < a_k = b$ in (a, b) such that, in each sub-interval (a_j, a_{j+1}) the function f is either constant, or strictly monotone and a continuous bijection with an interval.*

This theorem can be derived from the following three lemmas.

Lemma 2.1.4. *Let $\mathcal{M}$ be an o-minimal structure and $f : I \to M$ a definable function, for some open interval I. Then there is an open sub-interval of I where f is either constant or injective.*

Proof. Assume that there exists $x \in M$ such that $f^{-1}(\{x\})$ is infinite. Because pre-images of definable sets are definable, and $\mathcal{M}$ is o-minimal, $f^{-1}(\{x\})$ is a finite union of singletons and open intervals. Since $f^{-1}(\{x\})$ is infinite, there exists at least one open interval $J \subseteq I$ such that $J \subseteq f^{-1}(\{x\})$, i.e. $f|_J$ is constant.

On the other hand, assume that for all $x \in M$, $f^{-1}(\{x\})$ is finite. This implies that $f(I)$ is infinite, and by o-minimality and the fact that the image of definable sets under a definable function is definable, there exists an open interval J such that $J \subseteq f(I)$. Consider the following function $g : J \to I$ given by

$$g(y) = \max\{x \in I : f(x) = y\}$$

which is definable, and because it is injective, $g(J)$ is infinite. By the same argument as before, there exists an open sub-interval $K \subseteq I$ such that $K \subseteq g(J)$, and $f|_K$ is injective. $\square$

Lemma 2.1.5. *Let $\mathcal{M}$ be an o-minimal structure and $f : I \to M$ a definable function, for some open interval I. If f is injective, then f is strictly monotone on an open sub-interval of I.*

Proof. Write $I = (a,b)$ and assume that f is injective. Start by noting that, for each $x \in (a,b)$, we can partition the interval (a,x) as:

$$(a,x) = \underbrace{\{y \in (a,x) : f(y) < f(x)\}}_{A} \cup \underbrace{\{y \in (a,x) : f(y) > f(x)\}}_{B}$$

Both A and B are definable, and because (a,x) is infinite, at least one of them is infinite. This implies that one of them contains an interval of the form (c,x) for some $a < c < x$. This is due to the fact that:

If A is infinite and B finite, then A contains (c,x) for some $c > \max B$, and the same argument goes if B is infinite and A finite.

On the other hand, assume that both A and B are infinite, and assume that neither A nor B contain an interval of the form (c,x). This means that both A and B have elements arbitrarily close to x in (a,b). However, this is impossible, since by o-minimality A and B are finite unions of points and open intervals.

The interval (x,b) can be partitioned in a similar way. This means that each $x \in I$ satisfies exactly one of the following formulas:

$$\begin{aligned}\phi_{++}(x) := \exists c_1, c_2 \in I \Big(c_1 < x < c_2 \wedge \forall y \in (c_1,x) \ \big(f(y) > f(x)\big) \\ \wedge \forall y \in (x,c_2) \ \big(f(y) > f(x)\big)\Big)\end{aligned}$$

$$\begin{aligned}\phi_{+-}(x) := \exists c_1, c_2 \in I \Big(c_1 < x < c_2 \wedge \forall y \in (c_1,x) \ \big(f(y) > f(x)\big) \\ \wedge \forall y \in (x,c_2) \ \big(f(y) < f(x)\big)\Big)\end{aligned}$$

$$\begin{aligned}\phi_{--}(x) := \exists c_1, c_2 \in I\Big(c_1 < x < c_2 \wedge \forall y \in (c_1, x)\ \big(f(y) < f(x)\big) \\ \wedge \forall y \in (x, c_2)\ \big(f(y) < f(x)\big)\Big)\end{aligned}$$

$$\begin{aligned}\phi_{-+}(x) := \exists c_1, c_2 \in I\Big(c_1 < x < c_2 \wedge \forall y \in (c_1, x)\ \big(f(y) < f(x)\big) \\ \wedge \forall y \in (x, c_2)\ \big(f(y) > f(x)\big)\Big)\end{aligned}$$

Where $\exists c_1, c_2 \in I \ \ (\psi(c_1, c_2))$ is short for $\exists c_1 \exists c_2 (a < c_1 < b \wedge a < c_2 < b \wedge \psi(c_1, c_2))$ and analogously $\forall y \in (c_1, x) \ \ (\psi(y))$ is short for $\forall y((c_1 < y < x) \rightarrow \psi(y))$.

Consider the sets

$$A_{\square\triangle} = \{x \in M : \mathcal{M} \models (a < x < b \wedge \phi_{\square\triangle}(x))\}$$

for $\square, \triangle \in \{+, -\}$, which are definable. Note now that we can write I as a disjoint union:

$$I = A_{++} \cup A_{+-} \cup A_{--} \cup A_{-+}$$

Thus, one of these sets is infinite, and by o-minimality, there is a sub-interval $(c, d) \subseteq A_{\square\triangle}$ for some $\square, \triangle \in \{+, -\}$. We will prove that f is strictly monotone either in (c, d) or in some sub-interval of (c, d) by dividing the proof into 4 cases.

Case 1: $(c, d) \subseteq A_{-+}$.

This means that for all $x \in (c, d)$ we have that $\mathcal{M} \models \phi_{-+}[x]$. For each $x \in (c, d)$, define

$$s(x) = \sup\{y \in (x, d) : f > f(x) \text{ on } (x, y]\}$$

Then $s(x) = d$, because $s(x) < d$ would contradict the fact that $\mathcal{M} \models \phi_{-+}[s(x)]$. Now, note that given $\alpha < \beta$ in (c, d), then $s(\alpha) = d$ implies that $f(\alpha) < f(\beta)$ by definition of $s(\alpha)$, and thus f is strictly increasing.

Case 2: $(c, d) \subseteq A_{-+}$.

Using a similar argument as in case 1 we get that f is strictly decreasing in (c, d).

Case 3: $(c, d) \subseteq A_{++}$.

Consider the definable set

$$B = \{x \in (c,d) : \forall y \in (c,d)(y > x \rightarrow f(y) > f(x))\}$$

If B is infinite, by o-minimality, there is some open interval contained in B and f would be strictly increasing on that interval.

On the other hand, assume that B is finite. By considering a sub-interval of (c,d) to the right of all points of B, we may assume that

$$\forall x \in (c,d)\ \exists y \in (c,d)\ (y > x \wedge f(y) < f(x)) \qquad (*)$$

Claim: Let $k \in (c,d)$. Then, for large enough y in (c,d), we have that $f(y) < f(k)$.

Let

$$A = \{y \in (k,d) : f(y) < f(k)\}$$

$$B = \{y \in (k,d) : f(y) > f(k)\}$$

which are both definable. In particular, note that (k,d) is the disjoint union of A and B while means that there exists $l \in (k,d)$ such that either $(l,d) \subseteq A$ or $(l,d) \subseteq B$, and note that the claim we are proving is equivalent to saying that $(l,d) \subseteq A$.

For the sake of contradiction, assume that $(l,d) \subseteq B$ and let w be minimal in (k,d) such that $(w,d) \subseteq B$. Note that as $\mathcal{M} \models \phi_{++}[w]$, if $f(w) > f(k)$, w would not be minimal with respect to its defining property, therefore $f(w) < f(k)$. By $(*)$ there exists $e \in (w,d)$ such that $f(e) < f(w) < f(k)$ which is a contradiction, so we conclude that for large enough y in (c,d), $f(y) < f(k)$.

Let w_k be the least element of $[k,d)$ such that for any $y \in (w_k,d)$ we have $f(y) < f(k)$. Note that $\mathcal{M} \models \phi_{++}[k]$ implies that $k < w_k$ and $\mathcal{M} \models \phi_{++}[w_k]$ implies $f(w_k) < f(k)$, since otherwise w_k would not be minimal.

Consider now the formula

$$\psi_{>}(v) = \exists v_1, v_2 \in (c,d)\Big(v_1 < v < v_2 \wedge \forall z_1 \forall z_2 \big(v_1 < z_1 < v < z_2 < v_2 \rightarrow f(z_1) > f(z_2)\big)\Big)$$

The minimality of w_k implies that $\mathcal{M} \models \psi_{>}[w_k]$.

As k was chosen arbitrarily in (c,d), we have proved that

$$\forall x \in (c,d) \exists z \in (c,d)(x < z \wedge \psi_{>}(z))$$

This means that the set $\{x \in M : \mathcal{M} \models \psi_{>}[x]\}$ is infinite, and by o-minimality, there is an interval $(e,f) \subseteq (c,d)$ where $\psi_{>}$ hols. Consider now the formula:

$$\psi_{<}(v) = \exists v_1, v_2 \in (c,d)(v_1 < v < v_2 \wedge \forall z_1 \forall z_2 (v_1 < z_1 < v < z_2 < v_2 \rightarrow f(z_1) < f(z_2)))$$

A similar argument shows that there is a sub-interval of (e,f) where $\psi_{<}$ holds for all elements, which contradicts the fact that $\psi_{>}$ hols in all elements of (e,f).

With this, we see B being finite is impossible.

Case 4: $(c,d) \subseteq A_{--}$.

This is impossible for the same reason as case 3. $\square$

Lemma 2.1.6. *Let $\mathcal{M}$ be an o-minimal structure and $f : I \rightarrow M$ a definable function, for some open interval I. If f is strictly monotone, then f is continuous on an open sub-interval of I.*

Proof. Assume that f is strictly increasing. As any strictly increasing function is injective, $f(I)$ is infinite, and by o-minimality, there is an open interval $J \subseteq f(I)$. Take $c, d \in J$ with $c < d$ and let $a, b \in I$ such that $f(a) = c$ and $f(b) = d$, with $a < b$.

We start by proving that $f|_{(a,b)} : (a,b) \rightarrow (c,d)$ is bijective.

Injectivity: $f|_{(a,b)}$ is the restriction of an injective function.

Surjectivity: Let $e \in (c,d) \subseteq J$. As $J \subseteq f(I)$, there exists $k \in I$ such that $f(k) = e$. From the fact that $c < e < d$ and that f is strictly increasing, we get that $a < k < b$, and thus $k \in (a,b)$.

Given any open sub-interval $(\alpha, \beta) \subseteq (c,d)$, we have that $f^{-1}((\alpha,\beta)) = (f^{-1}(\alpha), f^{-1}(\beta))$, and so f is continuous in (a,b).

In the case where f is strictly increasing, the proof is analogous. $\square$

Equipped with these lemmas we can prove the Monotonicity theorem.

Proof. Consider the definable set X consisting of all points x in (a, b) such that, in some sub-interval of (a, b) containing x, the function f is either constant or strictly monotone and continuous. In particular, note that the set $(a, b) \setminus X$ is definable. If $(a, b) \setminus X$ were infinite, by o-minimality, there would exist an interval $I \subseteq (a, b) \setminus X$. Using lemmas 2.1.4,2.1.5 and 2.1.6 we can find a sub-interval $J \subseteq I$ where f is either constant or strictly monotone and continuous, meaning that $J \subseteq X$, which contradicts the fact that $J \subseteq I \subseteq (a, b) \setminus X$.

From this, we conclude that $(a, b) \setminus X$ is finite. Because we want to find a partition of (a, b) on which f is either constant or strictly monotone and continuous, it is enough to find such partition for each interval in X, so we may assume that $(a, b) = X$ and in particular f is continuous. Now, note that we can further partition (a, b) into disjoint definable sets:

$$(a, b) = A_1 \cup A_2 \cup A_3$$

where:

- $A_1 = \{x \in (a, b) : f \text{ is constant in some neighborhood of } x\}$
- $A_2 = \{x \in (a, b) : f \text{ is strictly increasing in some neighborhood of } x\}$
- $A_3 = \{x \in (a, b) : f \text{ is strictly decreasing in some neighborhood of } x\}$

Because they are definable, by o-minimality, each of them is the union of a finite number of intervals and a finite set, meaning they induce a finite partition of (a, b) where each sub-interval defined by the partition is contained in some A_i. It is enough then, to prove that in each sub-interval, f is either constant or strictly monotone, so assume that $(a, b) = A_i$ for some i.

Case 1: $(a, b) = A_1$

Let $x \in (a, b)$ and define

$$s = \sup\{k \in (x, b) : f \text{ is constant in } [x, k)\}$$

If $s < b$, because $s \in A_1$, there would exist an interval (c_1, c_2) containing s where f is constant, meaning that f would be constant in $[x, c_1)$, contradicting the definition of s. So $s = b$ and thus f is constant in $[x, b)$. An analogous argument shows that f is constant in $(a, x]$, and therefore constant in (a, b).

Case 2: $(a,b) = A_2$

Let $x \in (a,b)$ and define

$$s = \sup\{k \in (x,b) : f \text{ is strictly increasing in } [x,k)\}$$

Then $s = b$ by the same argument we did in case 1 and therefore f is strictly increasing in $[x,b)$. Again, by an analogous argument, we can show that f is strictly increasing in $(a,x]$ and thus strictly increasing in (a,b).

Case 3: $(a,b) = A_3$

The proof is analogous to Case 2. $\square$

Note that, if $f : (a,b) \to M$ is A-definable, then the points $a = a_0 < \ldots < a_k = b$ obtained from the Monotonicity Theorem are in fact also A-definable.

2.1.2 Cell Decomposition

The definition of o-minimality tells us what the definable sets of M look like, and the monotonicity theorem determines the behaviour of definable 1-ary functions on M. The next natural step is to determine what the definable sets in M^n are, and how definable functions in M^n behave. This culminates in the cell decomposition theorem which we will explore in this subsection.

For any definable subset $X \subseteq M^n$ let

$$C(X) := \{f : X \to M : f \text{ is continuous and definable}\}$$

and let $C_\infty(X) = C(X) \cup \{-\infty, \infty\}$, where $-\infty, \infty$ represent the constant functions with value $-\infty$ and ∞ respectively.

For any $f, g \in C_\infty(X)$, write $f < g$ if $f(x) < g(x)$ for all $x \in X$. If this is the case, we define the set:

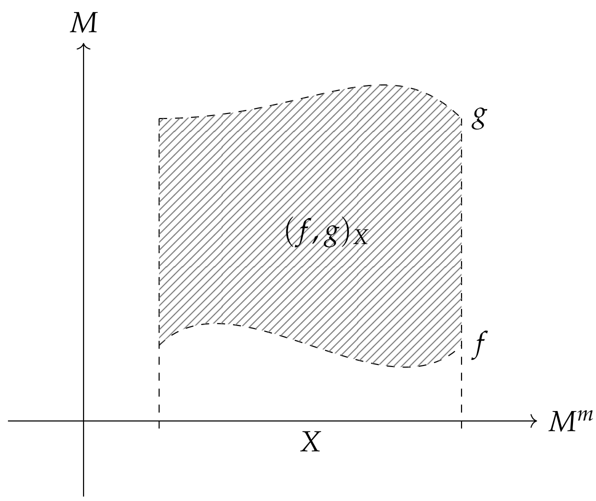
$$(f,g)_X = \{(\bar{x}, y) \in X \times M : f(\bar{x}) < y < g(\bar{x})\}$$

which is a definable subset of M^{n+1}. When it is clear from context what the set X is, we simply write (f,g).

With these definitions in mind, I will introduce the definition of a cell. As the name suggests, cells are the central object we use to characterize definable sets in M^n in the cell decomposition theorem.

Definition 2.1.7. Let $(i_1, \ldots, i_n)$ be a sequence of zeros and ones. An $(i_1, \ldots, i_n)$-*cell* is a definable subset of M^n defined recursively as follows:

1. A (0)-cell is simply a singleton of M, and a (1)-cell is a non-empty open interval of M (possibly unbounded);
2. Suppose that $(i_1, \ldots, i_n)$-cells are already defined:
   (a) An $(i_1, \ldots, i_n, 0)$-cell is the graph $\Gamma(f)$, for some $f \in C(X)$ where X is some $(i_1, \ldots, i_n)$-cell;
   (b) An $(i_1, \ldots, i_n, 1)$-cell is a set of the form $(f, g)_X$ where X is a $(i_1, \ldots, i_n)$-cell and $f, g \in C_\infty(X)$ with $f < g$. (see figure)



We say that a subset $X \subseteq M^n$ is a cell if it is a $(i_1, \ldots, i_n)$-cell, for some $i_k \in \{0, 1\}$.

Before continuing, I would like to make some remarks regarding this definition.

1. The numbers $i_1, \ldots, i_n$ are uniquely determined by the cell;
2. A cell is open in M^n if and only if it is a $(1, \ldots, 1)$-cell, and for this reason we refer to them as open cells.
3. Every cell is homeomorphic to an open cell under a projection. Let X be an $(i_1, \ldots, i_n)$-cell in M^n, let $k = i_1 + \ldots + i_n$ and let $\lambda(1) < \ldots < \lambda(k)$ such that $i_{\lambda(1)} = \ldots =$

$i_{\lambda(k)} = 1$. Then the map $\pi : M^n \to M^k$ projecting onto the $\lambda(1), \ldots, \lambda(k)$ coordinates is an homeomophism;

4. If X is a cell in M^{n+1}, then the image of X under the projection map into the fist n coordinates is a cell in M^n.

As the name suggests, the Cell Decomposition Theorem is formulated using the concept of a decomposition, which I introduce now.

Definition 2.1.8. A *decomposition* of M^n is a partition of M^n using finitely many cells, defined recursively as follows.

i) Any partition of M into finitely many cells is a decomposition;

ii) Let $\pi : M^{n+1} \to M^n$ be the projection onto the fist n coordinates. A decomposition of M^{n+1} is a partition of M^{n+1} into finitely many cells, such that the image under π of all cells is a decomposition of M^n.

One way to think about decompositions is the following:

Let $\mathcal{D} = (A_1, \ldots, A_k)$ be a decomposition of M^n, and for each i let $f_{i,1} < \ldots < f_{i,n_i}$ be functions in $C(A_i)$.

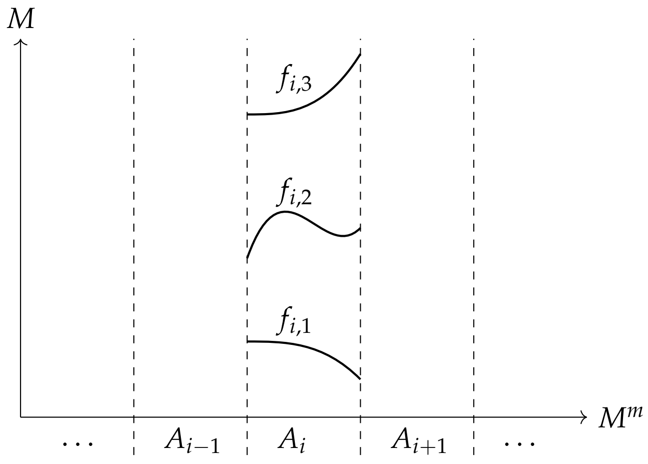
The set

$$\mathcal{D}_i = \{(-\infty, f_{i,1}), (f_{i,1}, f_{i,2}), \ldots, (f_{i,n_i}, \infty)\} \cup \{\Gamma(f_{i,1}), \ldots, \Gamma(f_{i,n_i})\}$$

is a partition of $A_i \times M$. Then the set

$$\mathcal{D}^* = D_1 \cup \ldots \cup D_k$$

is a decomposition of M^{n+1}, and in particular, every decomposition of M^{n+1} arises from a decomposition of M^n in this way. (see figure.)



Let X be a subset of M^n. We say that a decomposition of M^n partitions X if every cell in the decomposition is either contained in X or disjoint from X, or equivalently, if X is the disjoint union of cells from the decomposition. If $\mathcal{D}$ is a decomposition of M^n that partitions X, it is common to refer to the cells contained in X as a decomposition of X.

Theorem 2.1.9 (Cell Decomposition Theorem)**.** *Let $\mathcal{M}$ be o-minimal. For each $n > 0$ the following hold.*

(I) Given any $A_1, \dots, A_k$ definable subsets of M^n, there exists a decomposition of M^n that partitions each A_i;

(II) Given any definable function $f : A \to M$ where $A \subseteq M^n$ is definable, there is a decomposition $\mathcal{D}$ of M^n that partitions A, such that the restriction $f|_C : C \to M$ is continuous, for each $C \in \mathcal{D}$ with $C \subseteq A$;

(III) Given a formula $\phi(v, w_1, \dots, w_n)$ such that for all $\bar{a} \in M^n$, $\phi(\mathcal{M}, \bar{a})$ is finite, then there exists $N \in \mathbb{N}$ such that, for all $\bar{a} \in M^n$, $|\phi(\mathcal{M}, \bar{a})| \leq N$.

Proof. I will only sketch the proof. For more details see [8].

The idea is to prove (I), (II) and (III) at the same time using induction. Let (I)$_i$, (II)$_i$ and (III)$_i$ denote statements (I),(II) and (III) when we take $n = i$. (I)$_1$ holds by the definition of o-minimality, and (II)$_1$ follows from the monotonicity theorem. (III)$_1$ requires a direct argument and does not follow directly from any theorems I have stated before. The induction is done as follows:

1. (I)$_m$, (II)$_m$, (III)$_m$ (for $m < n$) $\implies$ (I)$_n$;

2. $(\text{I})_m$ (for $m \leq n$), $(\text{II})_m$ (for $m < n$) $\implies$ $(\text{II})_n$;

3. $(\text{I})_m$, $(\text{II})_m$ (for $m \leq n$) and $(\text{III})_m$ (for $m < n$) $\implies$ $(\text{III})_n$; □

Before continuing I would like to make the following remarks regarding Theorem 2.1.9:

- In (I), we may choose the decomposition $\mathcal{D}$ of M^n that partitions each $A_i, \ldots, A_k$ such that each cell in $\mathcal{D}$ is actually definable over the same parameters used to define $A_1, \ldots, A_k$, i.e. if each A_i is B_i-definable, we may assume that the cells in $\mathcal{D}$ are $\bigcup_{i=1}^k B_i$-definable. In particular, if we only have one set $X \subseteq M^n$, we can always find a decomposition of M^n that partitions X such that each cell is defined over the same parameters as X.
- Property (III) is called uniform boundedness, and a structure $\mathcal{M}$ that satisfies (III) for all $n \in \mathbb{N}$ is said to be uniformly bounded.

In particular, we have shown that every o-minimal structure is uniformly bounded.

Corollary 2.1.10. *Any o-minimal structure is uniformly bounded.*

From this, we can prove that being o-minimal is a property preserved under elementary equivalence.

Theorem 2.1.11. *Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures such that $\mathcal{M} \equiv \mathcal{N}$. If $\mathcal{M}$ is o-minimal, then $\mathcal{N}$ is o-minimal.*

Proof. Let $A \subseteq N$ be a definable set equal to $\phi(\mathcal{N}, \bar{b})$ for some $\phi(v, \bar{w})$ and $b \in N^n$.

Consider the following formulas:

$$\begin{aligned} pt(v,\bar{w}) =& \phi(v,\bar{w}) \wedge \forall c_1 \forall c_2 \Big(c_1 < v < c_2 \\ & \rightarrow \exists z_1 \exists z_2 \Big(c_1 < z_1 < v < z_2 < c_2 \wedge \neg\phi(z_1,\bar{w}) \wedge \neg\phi(z_2,\bar{w}) \Big) \Big) \end{aligned}$$

$$sup(v,\bar{w}) = \exists c_1 \forall z_1 \Big(c_1 < z_1 < v \rightarrow \phi(z_1,\bar{w}) \Big) \wedge \forall c_2 \exists z_2 \Big(v < z_2 < c_2 \wedge \neg\phi(z_2,\bar{w}) \Big)$$

$$inf(v,\bar{w}) = \forall c_1 \exists z_1 \Big(c_1 < z_1 < v \wedge \neg\phi(z_1,\bar{w}) \Big) \wedge \exists c_2 \forall z_2 \Big(v < z_2 < c_2 \rightarrow \phi(z_2,\bar{w}) \Big)$$

Now consider the formula

$$\psi(v,\bar{w}) = pt(v,\bar{w}) \vee sup(v,\bar{w}) \vee inf(v,\bar{w})$$

By o-minimality, for any $\bar{a} \in M^n$, the set $\phi(\mathcal{M},\bar{a})$ is a finite union of points open intervals, and thus $\psi(\mathcal{M},\bar{a})$ is finite. By uniform boundedness, there exists $K \in \mathbb{N}$ such that $|\psi(\mathcal{M},\bar{a})| < K$, for all $\bar{a} \in M^n$. In other words:

$$\mathcal{M} \models \forall\bar{w}\ \exists^{<K}v\ \psi(v,\bar{w})$$

As $\mathcal{N}$ is elementary equivalent to $\mathcal{M}$, we conclude that

$$\mathcal{N} \models \exists^{<K}v\ \psi(v,\bar{w})[\bar{b}]$$

This implies that $\phi(\mathcal{N},\bar{w}) = A$ is a finite union of open intervals and points. $\square$

Before continuing, I will state and prove some topological properties of cells that will be useful later on.

Recall that in a topological space X, we say that $A \subseteq X$ is locally closed if X is the intersection of an open and closed set. This is equivalent to the property that for any $x \in A$, there exists an open neighborhood U_x of x such that $U_x \cap A$ is closed in U_x. In particular, any open or closed set is locally closed.

I will start by proving the following lemma.

Lemma 2.1.12. *Let X, Y be topological spaces with Y being Hausdorff. Let $A \subseteq X$ be a locally closed set and $f : A \to Y$ be a continuous function. Then, $\Gamma(f)$ (the graph of f) is locally closed in $X \times Y$.*

Proof. Let $(x, f(x)) \in \Gamma(f)$. As A is locally closed, there exists $U_x \subseteq X$ open such that $x \in U_x$ and $U_x \cap A$ is closed in U_x. Consider the neighborhood $U_x \times Y$ of $(x, f(x))$. The goal is to show that $(U_x \times Y) \cap \Gamma(f)$ is closed in $U_x \times Y$. This follows from the fact that

$$(U_x \times Y) \cap \Gamma(f) = \{(x, f(x)) : x \in U_x \cap A\} = \Gamma(g)$$

where $g : U_x \cap A \to Y$ is the restriction of f to $U_x \cap A$. As $U_x \cap A$ is closed in U_x and Y is Hausdorff, from topology, we know that $\Gamma(g)$ is closed in $U_x \times Y$. $\square$

Proposition 2.1.13. *Let $\mathcal{M}$ be o-minimal. Then any cell $C \subseteq M^n$ is locally closed.*

Proof. The proof will be done using induction over n. For $n = 1$ the result is trivial. Now, let $C \subseteq M^{n+1}$ be a $(i_1, \ldots, i_n, k)$-cell. Start by assuming that $k = 0$, i.e. there exists a continuous definable function $f : C' \to M$ such that $C = \Gamma(f)$ (the graph of f), for some $(i_1, \ldots, i_n)$-cell C'. By the induction hypothesis, C' is locally closed, and by Lemma 2.1.12, we get that $\Gamma(f) = C$ is locally closed.

On the other hand, assume that $k = 1$, i.e. there exists an $(i_1, \ldots, i_n)$-cell C' and functions $f, g \in C_\infty(C')$ with $f < g$ such that $C = (f, g)_{C'}$. Let $(x, y) \in C = (f, g)_{C'}$, i.e. $x \in C'$ and $f(x) < y < g(x)$. As C' is locally closed, there exists an open neighborhood $U_x \subseteq M^n$ of x such that $U_x \cap C'$ is closed in U_x. Consider the open neighborhood $U_x \times M$ of (x, y). I claim that $(U_x \times M) \cap (f, g)_{C'}$ is closed in $U_x \times M$. Start by noting that

$$(U_x \times M) \cap (f, g)_{C'} = (f, g)_{U_x \cap C'}$$

Let $(a_i, b_i)_{i \in I}$ be a net in $(f, g)_{U_x \cap C'}$ that converges to (a, b) in $U_x \times M$. it is enough to show that $(a, b) \in (f, g)_{U_x \cap C'}$. Note that $(a_i)_{i \in I}$ is a net in $U_x \cap C'$ that converges to a in U_x, and as $U_x \cap C'$ is closed in U_x, we get that $a \in U_x \cap C'$. For each $i \in I$, we have that $f(a_i) < b_i < g(a_i)$ as $(a_i, b_i) \in (f, g)_{U_x \cap C'}$, which implies that $f(a) < b < g(a)$ and thus $(a, b) \in (f, g)_{U_x \cap C'}$. □

Another important property is definable connectedness, which generalizes the familiar notion of connectedness in a topological space.

Definition 2.1.14. Let $\mathcal{M} = (M, <, \ldots)$ be a linearly ordered structure and $X \subseteq M^n$. We say that X is *definably disconnected* if there are definable open subsets $U_1, U_2 \subseteq M^n$ such that $X \subseteq U_1 \cup U_2$, with $X \cap U_1, X \cap U_2$ non-empty and $X \cap U_1 \cap U_2 = \emptyset$.

If X is not definably disconnected, we say that X is *definably connected*.

Note that if X is definable, then X being definably disconnected is equivalent to saying that there are open (with the induced topology) disjoint non-empty definable subsets $U_1, U_2 \subseteq X$ such that $U_1 \cup U_2 = X$. In particular, X is definable disconnected if and only if X has a definable clopen subset, with respect to the induced topology. In particular, note that this implies that the image of a definably connected definable set under a definable continuous map is also definably connected.

Proposition 2.1.15. *Let $\mathcal{M}$ be o-minimal and $C \subseteq M^n$ be a cell. Then C is definably connected.*

For the proof, see Proposition 2.4 in [8].

2.1.3 Geometric structures and Dimension

In this subsection, we will explore two possible ways of developing the notion of dimension in an o-minimal structure.

Before continuing I will introduce the following notation: Given $A \subseteq M$ and a tuple $\bar{a} = (a_1, \ldots, a_n) \in M^n$, let $A\bar{a}$ denote $A \cup \{a_1, \ldots, a_n\}$.

We start by defining what a geometric structure is, and as we will see, o-minimal structures are one instance of such structures.

Definition 2.1.16. Let X be a non-empty set and $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ be a map. We say that (X, cl) is a *pregeometry* if it satisfies the following.

i) For all $A \subseteq X$, $A \subseteq \mathrm{cl}(A)$;

ii) For all $A \subseteq X$, $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;

iii) For all $A \subseteq X$,
$$\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(F) : F \text{ is a finite subset of } A\}$$

iv) (Exchange property) If $A \subseteq X$ and $b, c \in X$ with $b \in \mathrm{cl}(Ac) \setminus \mathrm{cl}(A)$, then $c \in \mathrm{cl}(Ab)$.

In any arbitrary structure $\mathcal{M}$ (not necessarily o-minimal), acl satisfies (i)-(iii). Furthermore, as we will see now, in any o-minimal structure, acl also satisfies the exchange property.

Theorem 2.1.17. *Let $\mathcal{M} = (M, <, \ldots)$ be an o-minimal structure. Then (M, acl) is a pregeometry.*

Proof. It is enough to show that acl satisfies the exchange property. By Proposition 2.0.2, it is enough to show that for all $A \subseteq M$ and $b, c \in M$,

$$b \in \mathrm{dcl}(Ac) \setminus \mathrm{dcl}(A) \implies c \in \mathrm{dcl}(Ab)$$

Furthermore, by adding the elements of A as new constants, it is enough to prove that

$$b \in \mathrm{dcl}(c) \setminus \mathrm{dcl}(\emptyset) \implies c \in \mathrm{dcl}(b)$$

Let $\{b\} = \{x \in M : \mathcal{M} \models \phi[x, c]\})$ for some formula $\phi(v, w)$.

Consider the set

$$A = \{x \in M : \mathcal{M} \models \exists^{=1} z \phi(z, v)[x]\}$$

This is an $\emptyset$-definable set with $c \in A$. If A is finite, then $c \in \text{acl}(\emptyset) \subseteq \text{acl}(b) = \text{dcl}(b)$ and we are done.

Assume then that A is infinite. By o-minimality, A is a finite union of open intervals and points. Consider now the formula

$$\psi(v) = v \in A \wedge \exists c_1 \exists c_2((c_1 \in A) \wedge (c_2 \in A) \wedge (c_1 < v < c_2) \wedge \forall z(c_1 < z < c_2 \rightarrow z \in A))$$

Where $v \in A$ is short for $\exists^{=1} z \phi(z, v)$. Then the set

$$B = \{x \in M : \mathcal{M} \models \psi[x]\}$$

is an $\emptyset$-definable set that consists of all points of A that lie inside an interval of A. On the other hand, the set $A \setminus B$ is also an $\emptyset$-definable subset of A consisting of all points that do not lie inside an interval of A.

Now, if $c \in A \setminus B$, then we would have that $c \in \text{acl}(\emptyset) \subseteq \text{acl}(b) = \text{dcl}(b)$ as this set is finite and we would be done.

On the other hand, consider that $c \in B$ and let (α, β) be the maximal interval containing c in A. I will start by showing that α and β are $\emptyset$-definable.

Again, by o-minimality, we can write A as:

$$A = (\alpha_1, \beta_1) \cup \ldots \cup (\alpha_k, \beta_k) \cup \{x_1, \ldots, x_w\}$$

Consider the formula

$$\begin{aligned}\psi(v) = \exists c_1((c_1 \in A) \wedge \forall z(c_1 < z < v \rightarrow z \in A)) \wedge \\ \forall c_2(c_2 > v \rightarrow \exists z(v < z < c_2 \wedge \neg(z \in A)))\end{aligned}$$

Then $\{\beta_1, \ldots, \beta_k\} = \{x \in M : \mathcal{M} \models \psi[x]\}$ is $\emptyset$-definable. So $\{\beta_1, \ldots, \beta_k\} \subseteq \text{acl}(\emptyset) = \text{dcl}(\emptyset)$. In an analogous way we conclude that each $\alpha_1, \ldots, \alpha_k$ is $\emptyset$-definable.

Now, consider the following $\emptyset$-definable function $f : (\alpha, \beta) \rightarrow M$, defined as follows:

$$(x, y) \in f \Leftrightarrow (\alpha < x < \beta) \wedge \phi(y, x)$$

Note that f maps $c \mapsto b$. By the monotonicity theorem, there exists $\alpha = a_1 < \ldots < a_k = \beta$ such that on each sub-interval f is either constant or strictly monotone.

If $c = a_i$ for some $i = 1, \ldots, k$, then c would be $\emptyset$-definable and we would be done. If this is not the case, then $c \in (a_i, a_{i+1})$ for some i.

If $f|_{(a_i, a_{i+1})}$ is constant, then it would map all elements to b as $f(c) = b$. Consider however the formula:

$$\psi(v) = \forall z(a_i < z < a_{i+1} \rightarrow f(z) = v)$$

Then we would have that $\{b\} = \{x \in M : \mathcal{M} \models \psi[x]\}$ and this would be a contradiction as b is not $\emptyset$-definable.

On the other hand, if $f|_{(a_i, a_{i+1})}$ is strictly monotone, then it is bijective, and its inverse $f|^{-1}_{(a_i, a_{i+1})}$ is $\emptyset$-definable. Because b itself is $\{b\}$-definable, the image $f|^{-1}_{(a_i, a_{i+1})}(b) = c$ is $\{b\}$-definable, thus concluding the proof. □

Definition 2.1.18. Let $\mathcal{M}$ be a first-order structure. We say that $\mathcal{M}$ is *geometric* if algebraic closure defines a pregeometry in any model of Th$(\mathcal{M})$, and $\mathcal{M}$ is uniformly bounded.

From Corollary 2.1.10, we have the following.

Corollary 2.1.19. *Every o-minimal structure is geometric.*

The fact that every o-minimal structure is geometric has very interesting consequences, in particular, there is a common way of assigning a dimension to definable sets in any geometric structure, similar to how dimension is defined in vector spaces.

Definition 2.1.20. Let $\mathcal{M}$ be a geometric structure, $A \subseteq M$ and $\bar{a} \in M^n$. We define $\dim(\bar{a}/A)$ as the least cardinality of a sub-tuple $\bar{a}'$ of $\bar{a}$ such that $\bar{a} \subseteq$ acl$(A\bar{a}')$.

If $p(\bar{x}) \in S_n^{\mathcal{M}}(A)$, we define $\dim(p)$ to be $\dim(\bar{a}/A)$, for any $\bar{a}$ realizing p in an elementary extension of $\mathcal{M}$.

Before continuing further, we will prove that the definition of $\dim(p)$ for $p \in S_n^{\mathcal{M}}(A)$ is well-defined.

Lemma 2.1.21. *Let $\mathcal{M}$ be geometric, $A \subseteq M$ and $p \in S_n^{\mathcal{M}}(A)$. Let $\mathcal{N}_1, \mathcal{N}_2$ be two elementary extensions of $\mathcal{M}$ and let $\bar{a} \in \mathcal{N}_1$, $\bar{b} \in \mathcal{N}_2$ realize p.*

Then $\dim(\bar{a}/A) = \dim(\bar{b}/A)$.

Proof. Let $\dim(\bar{a}/A) = k$ and $\dim(\bar{b}/A) = w$. Thus

$$k = \min\{|\bar{a}'| : \bar{a}' \subseteq \bar{a} \text{ and } \bar{a} \subseteq \text{acl}(A\bar{a}')\} \tag{2.1}$$

Let $\bar{a} = (a_1, \ldots, a_n)$ and, without loss of generality, assume that $\bar{a}' = (a_{n-k+1}, \ldots, a_n)$ is one of the least sub-tuples in (2.1). Let $n \in \mathbb{N}$ and for each $i = 1, \ldots, n-k$ let $\phi_i(x, y_1, \ldots, y_k)$ be an $\mathcal{L}_A$-formula such that:

$$\mathcal{N}_1 \models \underbrace{(\phi_i(x, y_1, \ldots, y_k) \wedge \exists^{<n} z\phi_i(z, y_1, \ldots, y_k))}_{:=\psi_i(x, y_1, \ldots, y_k)}[a_i, a_{n-k+1}, \ldots, a_n]$$

Then

$$\mathcal{N}_1 \models (\psi_1(x_1, y_1, \ldots, y_k) \wedge \ldots \wedge \psi_{n-k}(x_{n-k}, y_1, \ldots, y_k))\,[a_1, \ldots, a_{n-k}, a_{n-k+1}, \ldots, a_n]$$

As p is complete, we have that $(\psi_1(x_1, y_1, \ldots, y_k) \wedge \ldots \wedge \psi_{n-k}(x_{n-k}, y_1, \ldots, y_k)) \in p$ and thus:

$$\mathcal{N}_2 \models (\psi_1(x_1, y_1, \ldots, y_k) \wedge \ldots \wedge \psi_{n-k}(x_{n-k}, y_1, \ldots, y_k))\,[\bar{b}]$$

This means that $w \leq k$.

An analogous argument, starting with $\bar{b} \in N_2$, shows that $k \leq w$, and thus $k = w$. $\square$

The following lemma establishes some important properties, but before that I would like to introduce the concept of independent sets, which can be used to give an alternative definition of $\dim(\bar{a}/A)$.

Let $\mathcal{M}$ be any geometric structure, and let $I \subseteq M$. We say that I is independent if, for all $x \in I$, $x \notin \text{acl}(I \setminus \{x\})$. If $A \subseteq M$, we say that I is independent over A if it is independent when regarding the elements of A as new constant symbols.

Lemma 2.1.22. *Let $\mathcal{M}$ be a geometric structure, $A, B \subseteq M$ and $\bar{a}, \bar{b} \in M^n$. Then:*

i) $\dim(\bar{a}/A)$ *is equal to* $\max\{|\bar{a}'| : \bar{a}' \subseteq \bar{a}$ *is an independent sub-tuple over* $A\}$;

ii) *If $A \subseteq B$, then $\dim(\bar{a}/A) \geq \dim(\bar{a}/B)$;*

iii) $\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A\bar{b}) + \dim(\bar{b}/A)$;

iv) *If $p \in S_n^{\mathcal{M}}(A)$ and $A \subseteq B$, then there exists $p' \in S_n^{\mathcal{M}}(B)$ such that $p \subseteq p'$ and $\dim(p) = \dim(p')$.*

Proof. i) Start by adding the elements of A to the language as constant symbols so that, without loss of generality, we may assume that $A = \emptyset$.

We will start proving that $\bar{b} \subseteq \bar{a}$ is minimal such that $\bar{a} \subseteq \mathrm{acl}(\bar{b})$ if and only if $\bar{b}$ is a maximal independent sub-tuple of $\bar{a}$.

Start by assuming that $\bar{b}$ is a maximal independent sub-tuple of $\bar{a}$. I claim that $\bar{a} \subseteq \mathrm{acl}(\bar{b})$

To see this, assume for the sake of contradiction that $\bar{a} \not\subseteq \mathrm{acl}(\bar{b})$. Note that this implies that $\bar{a} \setminus \bar{b} \not\subseteq \mathrm{acl}(\bar{b})$, and so let $w \in \bar{a} \setminus \bar{b}$ such that $w \notin \mathrm{acl}(\bar{b})$.

Claim: $\bar{b}w$ is independent.

Let $b_i \in \bar{b}$ and note that if we had that $b_i \in \mathrm{acl}(\bar{b} \setminus \{ b_i \} \cup \{ w \})$ then

$$b_i \in \mathrm{acl}(\bar{b} \setminus \{ b_i \} \cup \{ w \}) \setminus \mathrm{acl}(\bar{b} \setminus \{ b_i \})$$

which, by the exchange property, would imply that $w \in \mathrm{acl}(\bar{b})$. Additionally, we already had established that $w \notin \mathrm{acl}(\bar{b})$, and therefore, $\bar{b}w$ is independent. However, this contradicts the maximality of $\bar{b}$.

Therefore, $\bar{a} \subseteq \mathrm{acl}(\bar{b})$. Now, assume that there exists a proper sub-tuple $\bar{c} \subset \bar{b}$ such that $\bar{a} \subseteq \mathrm{acl}(\bar{c})$, and let $b_i \in \bar{b} \setminus \bar{c}$. This would imply that $b_i \in \mathrm{acl}(\bar{c}) \subseteq \mathrm{acl}(\bar{b} \setminus \{b_i\})$, which contradicts the fact that $\bar{b}$ is independent, thus $\bar{b}$ is minimal with respect to the property that $\bar{a} \subseteq \mathrm{acl}(\bar{b})$.

On the other hand, let $\bar{b} \subseteq \bar{a}$ be minimal such that $\bar{a} \subseteq \mathrm{acl}(\bar{b})$. If $\bar{b}$ is not independent, then there exists $b_i \in \bar{b}$ such that $b_i \in \mathrm{acl}(\bar{b} \setminus \{b_i\})$. But then this would imply that $\mathrm{acl}(\bar{b}) \subseteq \mathrm{acl}(\bar{b} \setminus \{b_i\})$ and in particular, $\bar{a} \subseteq \mathrm{acl}(\bar{b} \setminus \{b_i\})$, contradicting the minimality of $\bar{b}$. Additionally, note that the existence of a sub-tuple $\bar{c}$ of $\bar{a}$ with $\bar{b} \subset \bar{c}$ is impossible for the following reason: If you take any $x \in \bar{c} \setminus \bar{b}$, then as $\bar{c}$ is independent, $x \notin \mathrm{acl}(\bar{c} \setminus \{x\})$, and in particular, $x \notin \mathrm{acl}(\bar{b})$. However, this contradicts the fact that $\bar{a} \subseteq \mathrm{acl}(\bar{b})$.

Next, we prove that any two minimal sub-tuples of $\bar{b}$ and $\bar{c}$ of $\bar{a}$ such that $\bar{a} \subseteq \mathrm{acl}(\bar{b})$ and $\bar{a} \subseteq \mathrm{acl}(\bar{c})$ have the same number of elements.

For that, we start by proving the following generalization of Steinitz Exchange Lemma:

Claim: Let $\bar{b} \subseteq \bar{a}$ be independent and $\bar{c} \subseteq \bar{a}$ be a sub-tuple such that $\bar{a} \subseteq \mathrm{acl}(\bar{c})$. Then:

- $|\bar{b}| \leq |\bar{c}|$;

- There is $\bar{c}' \subseteq \bar{c}$ with $|\bar{c}| = |\bar{c}| - |\bar{b}|$ such that $\bar{a} \subseteq \mathrm{acl}(\bar{b}\bar{c}')$.

Let $\bar{b} = (b_1, \ldots, b_m)$ and $\bar{c} = (c_1, \ldots, c_k)$. We will show that $m \leq k$ using induction on m.

If $m = 0$ then it is trivial that $m \leq k$, and we may take $\bar{c}'$ to be $\bar{c}$.

Assume now that the claim is true for $m-1$. We start by applying the induction hypothesis to $\bar{b}' = \bar{b} \setminus \{b_m\}$, i.e. there exists $\bar{c}' \subseteq \bar{c}$ with $|\bar{c}'| = |\bar{c}| - (m-1)$ such that $\bar{a} \subseteq \mathrm{acl}(\bar{b}'\bar{c}')$. This means that $\bar{b}_m \in \mathrm{acl}(\bar{b}'\bar{c}')$. Note that if, for all $c_i \in \bar{c}'$, we had that $b_m \in \mathrm{acl}(\bar{b}'\bar{c}' \setminus \{c_i\})$, then $b_m \in \mathrm{acl}(\bar{b}')$, which contradicts the fact that $\bar{b}$ is independent. So there exists $c_i \in \bar{c}'$ such that $b_m \notin \mathrm{acl}(\bar{b}'\bar{c}' \setminus \{c_i\})$, which we fix now. Note however, that this implies that

$$b_m \in \mathrm{acl}(\bar{b}'(\bar{c}' \setminus \{c_i\}) \cup \{c_i\}) \setminus \mathrm{acl}(\bar{b}'\bar{c}' \setminus \{c_i\})$$

And therefore, by the Exchange property, we conclude that $c_i \in \mathrm{acl}(\bar{b}\bar{c}' \setminus \{c_i\})$ In particular, note that $|\bar{c}' \setminus \{c_i\}| = |\bar{c}'| - 1 = |\bar{c}| - k$, furthermore, $\bar{b}' \subseteq \mathrm{acl}(\bar{b}\bar{c}' \setminus \{c_i\})$ and $\bar{c}' \subseteq \mathrm{acl}(\bar{b}\bar{c}' \setminus \{c_i\})$, which implies that $\bar{a} \subseteq \mathrm{acl}(\bar{b}\bar{c}' \setminus \{c_i\})$. As $\bar{b}\bar{c}' \setminus \{c_i\}$ was obtained from $\bar{b}'\bar{c}'$ by adding b_m and removing c_i, we conclude that $|\bar{b}\bar{c}' \setminus \{c_i\}| = |\bar{b}'\bar{c}'| = |\bar{c}|$, and in particular, $|\bar{b}| \leq |\bar{c}|$.

Now, let $\bar{b}$ and $\bar{c}$ be two minimal sub-tuples of of $\bar{a}$ such that $\bar{a} \subseteq \mathrm{acl}(\bar{b})$ and $\bar{a} \subseteq \mathrm{acl}(\bar{c})$. As we have already seen, this implies that $\bar{b}$ is independent, and therefore by the claim we just proved, $|\bar{b}| \leq |\bar{c}|$. An analogous argument shows that $|\bar{b}| \geq |\bar{c}|$.

In particular, this means that if $\bar{b} \subseteq \bar{a}$ is a minimal sub-tuple such that $\bar{a} \subseteq \mathrm{acl}(\bar{b})$, then $|\bar{b}| = \dim(\bar{a}/\emptyset)$. To conclude the proof, let $\max\{|\bar{a}'| : \bar{a}' \subseteq \bar{a}$ is an independent sub-tuple over $A\} = k$ and $\bar{b} \subseteq \bar{a}$ be an independent sub-tuple of $\bar{a}$ with $|\bar{b}| = k$, i.e. $\bar{b}$ is maximal among other independent sub-tuples. Then, as we have seen $\bar{b}$ is a minimal subset of $\bar{a}$ such that $\bar{a} \subseteq \mathrm{acl}(\bar{b})$, and therefore $k = |\bar{b}| = \dim(\bar{a}/\emptyset)$.

ii) If $A \subseteq B$, then for any sub-tuple $\bar{a}' \subseteq \bar{a}$, we have that $A\bar{a}' \subseteq B\bar{a}'$ and thus $\mathrm{acl}(A\bar{a}') \subseteq \mathrm{acl}(B\bar{a}')$. This gives us the following inclusion:

$$\{\bar{a}' : \bar{a}' \subseteq \bar{a} \text{ and } \bar{a} \subseteq \mathrm{acl}(A\bar{a}')\} \subseteq \{\bar{a}' : \bar{a}' \subseteq \bar{a} \text{ and } \bar{a} \subseteq \mathrm{acl}(B\bar{a}')\}$$

and thus

$$\underbrace{\min\{|\bar{a}'| : \bar{a}' \subseteq \bar{a} \text{ and } \bar{a} \subseteq \mathrm{acl}(A\bar{a}')\}}_{=\dim(\bar{a}/A)} \geq \underbrace{\min\{|\bar{a}'| : \bar{a}' \subseteq \bar{a} \text{ and } \bar{a} \subseteq \mathrm{acl}(B\bar{a}')\}}_{=\dim(\bar{a}/B)}$$

Thus concluding the proof.

iii) Start by adding the elements of A to the language as constant symbols so that, without loss of generality, we may assume that $A = \emptyset$.

Let $\bar{a}' \subseteq \bar{a}$ be minimal such that $\bar{a} \subseteq \text{acl}(\bar{b}\bar{a}')$ and $\bar{b}' \subseteq \bar{b}$ minimal such that $\bar{b} \subseteq \text{acl}(\bar{b}')$. Then $\bar{a}\bar{b} \subseteq \text{acl}(\bar{a}'\bar{b}')$, meaning that

$$\dim(\bar{a}\bar{b}/\emptyset) \leq \dim(\bar{a}/\bar{b}) + \dim(\bar{b}/\emptyset)$$

On the other hand, let $\bar{c} = \bar{a}'\bar{b}' \subseteq \bar{a}\bar{b}$ minimal such that $\bar{a}\bar{b} \subseteq \text{acl}(\bar{c})$.

Then $\bar{a} \subseteq \text{acl}(\bar{a}'\bar{b}') \subseteq \text{acl}(\bar{b}\bar{a}')$, so $\dim(\bar{a}/\bar{b}) \leq |\bar{a}'|$. If $\bar{b} \subseteq \text{acl}(\bar{b}')$, then $\dim(b/\emptyset) \leq |\bar{b}'|$ and we would be done, as this would imply that

$$\dim(\bar{a}/\bar{b}) + \dim(\bar{b}/\emptyset) \leq |\bar{a}'\bar{b}'| = \dim(\bar{a}\bar{b}/\emptyset)$$

Assume now that $\bar{b} \not\subseteq \text{acl}(\bar{b}')$, i.e. there exists $b_i \in \bar{b} \setminus \bar{b}'$ such that $b_i \notin \text{acl}(\bar{b}')$.

Claim: There exists $a_j \in \bar{a}'$ such that if $\bar{c}'$ is obtained from $\bar{c}$ by removing a_j and adding b_i, then $|\bar{c}'| = |\bar{c}|$ and $\text{acl}(\bar{c}') = \text{acl}(\bar{c})$.

Let $\bar{a}'' \subseteq \bar{a}'$ be a minimal sub-tuple such that $b_i \in \text{acl}(\bar{a}''\bar{b}')$, which exists as $b_i \in \text{acl}(\bar{a}'\bar{b}')$. Because $b_i \notin \text{acl}(\bar{b}')$, we have that $\bar{a}'' \neq \emptyset$, and so, let $a_j \in \bar{a}''$ by any element and set $\bar{a}'''$ to be $\bar{a}'' \setminus \{a_j\}$. Note that $b_i \in \text{acl}(\bar{a}'''\bar{b}'a_j)$ while, by the minimality of $\bar{a}''$, $b_i \notin \text{acl}(\bar{a}'''\bar{b}')$, i.e.

$$b_i \in \text{acl}(\bar{a}'''\bar{b}'a_j) \setminus \text{acl}(\bar{a}'''\bar{b}')$$

By the exchange property, we conclude that $a_j \in \text{acl}(\bar{a}'''\bar{b}'b_i)$.

Let $\bar{c}'$ be obtained from $\bar{c}$ by removing a_j and adding b_i. Start by noting that $|\bar{c}'| = |\bar{c}|$ trivially. Additionally, as $\bar{a}\bar{b} \subseteq \text{acl}(\bar{c})$, we have that $\bar{c}' \subseteq \text{acl}(\bar{c})$ and therefore $\text{acl}(\bar{c}') \subseteq \text{acl}(\bar{c})$. On the other hand, note that $\bar{a}'''\bar{b}'b_j \subseteq \bar{c}'$ and thus $\text{acl}(\bar{a}'''\bar{b}'b_i) \subseteq \text{acl}(\bar{c}')$. However, note that we have already established by the exchange property that $a_j \in \text{acl}(\bar{a}'''\bar{b}'b_i)$, and therefore, $a_j \in \text{acl}(\bar{c}')$. Note also that as $\bar{b}' \subseteq \bar{c}'$ and $\bar{a}' \setminus \{a_j\} \subseteq \bar{c}'$, we have that $\bar{b}', \bar{a}' \setminus \{a_j\} \subseteq \bar{\text{acl}}(c')$. This implies that $\bar{c} = (\bar{a}' \setminus \{a_j\}) \cup \{a_j\} \cup \bar{b}' \subseteq \text{acl}(\bar{c}')$ and therefore $\text{acl}(\bar{c}) \subseteq \text{acl}(\bar{c}')$, completing the proof that $\text{acl}(\bar{c}) = \text{acl}(\bar{c}')$ and therefore proving the claim.

The idea now is to use the claim repeatedly in the following way:

Consider the sequence of pairs $(\bar{b}'_n, \bar{c}'_n)_{n\geq 0}$ defined as: If $n = 0$, then $(\bar{b}'_0, \bar{c}'_0) = (\bar{b}', \bar{c})$. If $(\bar{b}'_n, \bar{c}'_n)$ is already defined, then $(\bar{b}'_{n+1}, \bar{c}'_{n+1}) = (\bar{b}'_n, \bar{c}'_n)$ if $\bar{b} \subseteq$ acl$(\bar{b}'_n)$. On the other hand, if $\bar{b} \not\subseteq$ acl$(\bar{b}'_n)$, let $\bar{c}'_{n+1}$ be the sub-tuple of $\bar{a}\bar{b}$ with $|c'_{n+1}| = |c'_n|$ and acl$(c'_{n+1}) =$ acl(c'_n) obtained from applying the claim we just proved, and let $\bar{b}'_{n+1} = \bar{c}'_{n+1} \cap \bar{b}$.

Each step we apply the claim, we get a new sub-tuple $\bar{b}'_{n+1}$ with $|\bar{b}'_{n+1}| = |\bar{b}'_n| + 1$, and as $\bar{b}$ is finite, this means that we can only apply the claim a finite number of times. As a consequence of this, the sequence $(\bar{b}'_n, \bar{c}'_n)_{n\geq 0}$ eventually stabilizes. Let $(\bar{b}^*, \bar{c}^*)$ be the value that the sequence eventually takes, and note that $|\bar{c}^*| = |\bar{c}|$ and $\bar{b} \subseteq$ acl$(\bar{b}^*)$, i.e. $\dim(\bar{b}/\emptyset) \leq |\bar{b}^*|$. This allows us to conclude that:

$$\dim(\bar{a}/\bar{b}) + \dim(\bar{b}/\emptyset) \leq |\bar{a}'\bar{b}^*| = |\bar{c}^*| = |\bar{c}| = \dim(\bar{a}\bar{b}/\emptyset)$$

iv) We can expand $\mathcal{M}$ to a sufficiently saturated structure $\mathcal{N}$ where any type from $S_n^{\mathcal{M}}(A)$ and $S_n^{\mathcal{M}}(B)$ is realized. Therefore it is enough to prove the result for $\mathcal{N}$, so without loss of generality we may assume that $\mathcal{M}$ is sufficiently saturated.

Note that for any $p' \in S_n^{\mathcal{M}}(B)$, if $p \subseteq p'$, then $\dim(p) \geq \dim(p')$, as $\dim(\bar{a}'/A) \geq \dim(\bar{a}/B)$ for any $\bar{a}$ realizing p', so it is enough to find $p' \in S_n^{\mathcal{M}}(B)$ with $p \subseteq p'$ such that $\dim(p) \leq \dim(p')$.

Let $\dim(p) = k$ and for each $\mathcal{L}_B$-formula $\theta(v, w_1, \ldots, w_{k-1})$ and $n \in \mathbb{N}$, consider the formula $\Psi_{\theta,n}(v_1, \ldots, v_n)$ defined by

$$\bigwedge_{\sigma\in S(n)} \left(\neg\theta(v_{\sigma(n)}, v_{\sigma(1)}, \ldots, v_{\sigma(k-1)}) \vee \exists^{>n} z\, \theta(z, v_{\sigma(1)}, \ldots, v_{\sigma(k-1)})\right)$$

And consider the partial n-type over B

$$q = p \cup \bigcup_{\theta,n} \{\Psi_{\theta,n}\}$$

Note that this is indeed a type as Th$_B(\mathcal{M}) \cup q$ is finitely satisfiable, being witnessed by any realization of p as $\dim(p) = k$.

There exists $p' \in S_n^{\mathcal{M}}(B)$ such that $p \subseteq q \subseteq p'$, and the way we have defined each $\Psi_{\theta,n}$ guarantees that for any $\bar{a}$ realizing p', we have $\dim(\bar{a}/B) \geq k$. $\square$

Before continuing, I would like to introduce some notation. Let $X = \phi(\mathcal{M}, \bar{a})$ for some ϕ

and $\bar{a} \in A$. To simplify the notation, it is common to denote the $\mathcal{L}_A$-formula ϕ that defines X by X itself, i.e. it is common to write $X(\mathcal{M})$ for X. So given an elementary extension $\mathcal{M} \preceq \mathcal{N}$, it is usual to let $X(\mathcal{N})$ denote the set $\phi(\mathcal{N}, \bar{a})$. With this in mind, we can define dimensions for arbitrary definable sets in a geometric structure as follows.

Definition 2.1.23. Let $\mathcal{M}$ be a geometric structure and $\mathbb{M}$ be a sufficiently saturated elementary extension of $\mathcal{M}$. Given an A-definable set $X \subseteq M^n$, we define the (geometric) *dimension* of X

$$\dim(X) := \max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{M})\}$$

We say that $\bar{a} \in X$ is a generic of X over A if $\dim(\bar{a}/A) = \dim(X)$.

Note that if $\mathcal{M}$ itself is sufficiently saturated we can simply put $\dim(X) = \max\{\dim(\bar{x}/A) : \bar{x} \in X\}$

Before continuing further, we will prove that dimension is well-defined, i.e. it does not depend on the parameter set nor on which saturated elementary extension we choose.

Proposition 2.1.24. *Let $\mathcal{M}$ be a geometric structure and X be an A-definable subset. Then:*

i) If X is also B-definable then, fixing a sufficiently saturated elementary extension $\mathbb{M} \succeq \mathcal{M}$, we have

$$\max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{M})\} = \max\{\dim(\bar{x}/B) : \bar{x} \in X(\mathbb{M})\}$$

ii) If $\mathbb{M}, \mathbb{U} \succeq \mathcal{M}$ are two sufficiently saturated elementary extensions, then

$$\max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{M})\} = \max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{U})\}$$

Proof. Note that

$$\max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{M})\} = \max\{\dim(p) : p \in S_n^{\mathcal{M}}(A) \text{ and } p \text{ is realized on } X(\mathbb{M})\}$$

i) Let $\max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{M})\} = k$ and let $p \in S_n^{\mathcal{M}}(A)$ be a type such that $\dim(p) = k$ and p is realized in $X(\mathbb{M})$, i.e. there exists $\bar{a} \in X(\mathbb{M})$ such that $p = \text{tp}^{\mathcal{M}}(\bar{a}/A)$. By (iv) of lemma 2.1.27, there exists $q \in S_n^{\mathcal{M}}(A \cup B)$ such that $p \subseteq q$, $\dim(q) = k$, and by saturation we get that q is realized by some $\bar{b} \in \mathbb{M}^n$. In particular, $\bar{b}$ realizes $\text{tp}^{\mathcal{M}}(\bar{a}/A)$ meaning that $\text{tp}^{\mathcal{M}}(\bar{b}/A) = \text{tp}^{\mathcal{M}}(\bar{a}/A)$. Because $\mathbb{M}$ is homogeneous, there exists an automorphism $\sigma : \mathbb{M} \to \mathbb{M}$ such that $\sigma(\bar{a}) = \bar{b}$ and σ fixes A pointwise. As $X(\mathbb{M})$ is A-definable, σ fixes $X(\mathbb{M})$ set-wise, meaning that $\sigma(\bar{a}) = \bar{b} \in X(\mathbb{M})$.

Now note that $k = \dim(q) = \dim(\bar{b}/A \cup B) \leq \dim(\bar{b}/B)$ by (i) of Lemma 2.1.27, and thus

$$\max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{M})\} \leq \max\{\dim(\bar{x}/B) : \bar{x} \in X(\mathbb{M})\}$$

The other inequality can be proven analogously.

ii) Let $\mathbb{M}, \mathbb{U} \preceq \mathcal{M}$ be two sufficiently saturated elementary extensions of $\mathcal{M}$. If $|\mathbb{M}| = |\mathbb{U}|$, then $\mathbb{M} \simeq \mathbb{U}$, by Theorem 1.8.10 any two saturated models of the same cardinality are isomorphic, and the result follows. On the other hand, without loss of generality assume that $|\mathbb{M}| < |\mathbb{U}|$. As, by Corollary 1.8.6, $\mathbb{U}$ us universal, there is an elementary embedding $\mathbb{M} \hookrightarrow \mathbb{U}$, so we may as well assume that $\mathbb{M} \preceq \mathbb{U}$. In particular, $X(\mathbb{M}) \subseteq X(\mathbb{U})$, so that

$$\max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{M})\} \leq \max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{U})\}$$

On the other hand let $\max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{U})\} = k$, and $p \in S_n^{\mathcal{M}}(A)$ be a type such that $\dim(p) = k$, and there exists $\bar{a} \in X(\mathbb{U})$ that realizes p. By saturation there exists $\bar{b} \in \mathbb{M}^n$ that realizes p, meaning that $\mathrm{tp}^{\mathcal{M}}(\bar{a}/A) = \mathrm{tp}^{\mathcal{M}}(\bar{b}/A)$. Because, by Corollary 1.8.6 , $\mathbb{U}$ is homogeneous, there exists an automorphism $\sigma : \mathbb{U} \to \mathbb{U}$ such that $\sigma(\bar{a}) = \bar{b}$ and σ fixes A pointwise. The fact that σ fixes A pointwise implies that it fixes setwise all A-definable sets, and in particular $\sigma(\bar{a}) = \bar{b} \in X(\mathbb{U})$. As $\bar{b} \in \mathbb{M}^n$ and $X(\mathbb{U}) \cap \mathbb{M}^n = X(\mathbb{M})$ we get that p is a type with dimension k realized by $\bar{b} \in X(\mathbb{M})$, meaning that

$$\max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{M})\} \geq k = \max\{\dim(\bar{x}/A) : \bar{x} \in X(\mathbb{U})\}$$

Thus concluding the proof. □

This definition of dimensions has very interesting properties and, in particular, it is invariant under definable bijective maps. The following proposition is a collection of such properties that we will use in this work. For now, they are simply stated and we will prove them later.

Proposition 2.1.25. *Let $\mathcal{M}$ be a geometric structure. Then:*

i) $\dim(\{a\}) = 0$ *for all* $a \in M$ *and* $\dim(M) = 1$*;*

ii) *If* $A \subseteq B \subseteq M^n$ *are definable, then* $\dim(A) \leq \dim(B)$*;*

iii) If $X \subseteq M^n$ and $Y \subseteq M^k$ are definable, and there exists a definable bijection between X and Y, then $\dim(X) = \dim(Y)$;

iv) If $X \subseteq M^n$ and $Y \subseteq M^k$ are definable and $f : X \to Y$ is a definable injective map, then $dim(X) \leq \dim(Y)$;

v) If $X \subseteq M^n$ and $Y \subseteq M^k$ are definable, then $\dim(X \times Y) = \dim(X) + \dim(Y)$;

vi) If $X, Y \subseteq M^n$ are definable, then $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.

Although these properties are true in any geometric structure, when the structure is o-minimal, there is a different approach we can use to prove them, as we will see now. The key observation is that o-minimal structures have something special that not all geometric structures have: Cell decomposition. This gives rise to another alternative definition of dimension commonly known as topological dimension:

Definition 2.1.26. Let $\mathcal{M}$ be o-minimal. If C is an $(i_1, \ldots, i_n)$-cell, then we define

$$\text{tdim}(C) = i_1 + \ldots + i_n$$

Given a definable set $X \subseteq M^n$, we define:

$$\text{tdim}(X) = \max\{\text{tdim}(C) : C \subseteq X \text{ is a cell}\}$$

We also define tdim$(\emptyset)$ to be $-\infty$.

Note that in particular, if $X \subseteq M^n$ is definable, then tdim$(X) = n$ if and only if X contains an open cell. Also, it is clear from the definition that if $X \subseteq Y \subseteq M^n$ are definable, then tdim$(X) \leq$ tdim$(Y) \leq n$.

To prove some properties about this definition of dimension we will first need the following lemma.

Lemma 2.1.27. *Let $A \subseteq M^n$ be an open cell and $f : A \to M^n$ an injective definable map. Then $f(A)$ contains an open cell.*

Proof. We will use induction on n.

If $n = 1$ then A contains an open interval and so does $f(A)$, proving the result.

Now let $n > 1$ and assume that the result is true for lower values of n. By the cell decomposition theorem, there is a decomposition of M^n that partitions $f(A)$, so we can write

$$f(A) = C_1 \cup \ldots \cup C_k$$

and thus

$$A = f^{-1}(C_1) \cup \ldots \cup f^{-1}(C_k)$$

as A is open, at least one $f^{-1}(C_i)$ contains a box, and for the sake of simplicity assume that $f^{-1}(C_1)$ contains a box B. By the cell decomposition theorem, there is a decomposition that partitions B as

$$B = D_1 \cup \ldots \cup D_w$$

such that f is continuous in each D_i. By the same argument as before, some D_i contains a box and by taking B to be that smaller box inside that D_i, we have that $f^{-1}(C_1)$ contains a box B such that $f|_B : B \to C_1$ is continuous.

We will prove that C_1 is an open cell via a contradiction. Start by assuming then that C_1 is not open and for the sake of simplicity assume that C_1 is an $(i_1, \ldots, i_{n-1}, 0)$ - cell.

Consider the map $g : B \to M^{n-1}$ given by $g = p \circ f|_B$ where $p : C_1 \to M^{n-1}$ is the projection into the fist $n-1$ coordinates and in particular note that g is definable, continuous and injective.

Write B as $B' \times (a, b)$, let $c \in (a, b)$ and consider the map

$$\begin{aligned} h : B' &\to M^{n-1} \\ x &\mapsto g(x, c) \end{aligned}$$

By the induction hypothesis $h(B')$ contains an open cell in M^{n-1} and as such there is a box D in M^{n-1} such that $D \subseteq h(B')$. Let $x \in h^{-1}(D)$ be any element.

The fact that g is continuous implies that $g^{-1}(D) \subseteq B$ is open, and in particular there exists an open box K with $(x, c) \in K \subseteq g^{-1}(D)$. Writing K as $K' \times (\alpha, \beta)$ we see that for any $c' \in (\alpha, \beta)$ we have that $(x, c') \in K \subseteq g^{-1}(D)$, i.e. $g(x, c') \in D$. As $D \subseteq h(B')$, there exists $x' \in B'$ such that $g(x, c') = h(x')$ which contradicts the injectivity of g because $g(x, c') = h(x') = g(x', c)$. $\square$

With this lemma, we can prove the following.

Proposition 2.1.28. *Let $\mathcal{M}$ be o-minimal. Then:*

i) If $X \subseteq M^n$ and $Y \subseteq M^k$ are definable, and there exists a definable bijection between X and Y, then tdim$(X) =$ tdim(Y);

ii) If $X \subseteq M^n$ and $Y \subseteq M^k$ are definable and $f : X \to Y$ is a definable injective map, then tdim$(X) \leq$ tdim(Y) and tdim$(X) =$ tdim$(f(X))$;

iii) If $X, Y \subseteq M^n$ are definable, then tdim$(X \cup Y) = \max\{$tdim(X), tdim$(Y)\}$.

Proof. i) Let $f : X \to Y$ be a definable bijection. it is enough to prove that tdim$(X) \leq$ tdim(Y), because if this is the case, the reverse equality follows from the same argument but using f^{-1} instead of f. Let tdim$(X) = d$ and let A be an $(i_1, \dots, i_n)$-cell contained in X with tdim$(A) = d$. Let $p : M^n \to M^d$ be the map projection onto the coordinates for which i_k is not zero, so that $p(A)$ is an open cell in M^d. Consider the map

$$g = f \circ (p|_A)^{-1} : p(A) \to Y$$

We will prove that $im(g) = f(A)$ contains a cell of dimension d. To simplify the notation, replace X by $p(A)$, Y by $f(A)$ and f by $f \circ (p|_A)^{-1}$ (note that this is still a definable bijection) so that $n = d$ and X is an open cell.

Let $Y = C_1 \cup \ldots \cup C_k$ be a partition of Y into cells. As $Y = f(Y)$ this allows us to write:

$$X = f^{-1}(C_1) \cup \ldots \cup f^{-1}(C_k)$$

Again, by cell decomposition some $f^{-1}(C_i)$ contains an open cell. For the sake of simplicity assume that $f^{-1}(C_1)$ contains an open cell B.

Let C_1 be an $(j_1, \dots, j_m)$-cell. We will prove that $d \leq$ tdim(C_1). For this, assume that tdim$(C_1) < d$.

Consider the following maps.

$$f|_B : B \to C_1$$
$$p : C_1 \to p(C_1) \subseteq M^{\text{tdim}(C_1)}$$
$$q : p(C_1) \to M^d$$

Where:

p is the projection onto the coordinates for which j_k is not zero;

q maps $x \in p(C_1)$ to (x, w) for some fixed $w \in M^{d-\text{tdim}(C_1)}$.

The map $q \circ p \circ f|_B$ is an injective definable map from M^d to M^d and by Lemma 2.1.27 its image would contain an open cell. However, the image is contained in $M^{\text{tdim}(C_1)} \times \{w\}$ meaning that it can not contain any open cells.

So $d \leq \text{tdim}(C_1)$ and as $C_1 \subseteq Y$ we get that $d \leq dim(Y)$.

ii) Follows directly from i) and from the fact that if $A \subseteq B$ then $\text{tdim}(A) \leq \text{tdim}(B)$.

iii) As $X, Y \subseteq X \cup Y$, we have that $\text{tdim}(X), \text{tdim}(Y) \leq \text{tdim}(X \cup Y)$ and thus $\max\{\text{tdim}(X), \text{tdim}(Y)\} \leq \text{tdim}(X \cup Y)$.

On the other hand, without loss of generality assume that $\text{tdim}(X) \geq \text{tdim}(Y)$. Let $\text{tdim}(X \cup Y) = d$ and let A be an $(i_1, \ldots, i_n)$-cell contained in $X \cup Y$ such that $\text{tdim}(A) = d$. Consider the projection $p : A \to M^d$ onto the coordinates for which i_k is not zero, i.e. $p(A)$ is an open cell in M^d. Note that

$$p(A) = p(A \cap X) \cup p(A \cap Y)$$

By cell decomposition, at least one of them contains an open box and because $\text{tdim}(p(A \cap X)) \geq \text{tdim}(p(A \cap Y))$, we get that $p(A \cap X)$ contains such open box B. Note that $p^{-1}(B)$ is an $(i_1, \ldots, i_n)$-cell contained in X, meaning that $\text{tdim}(X) \geq d = \text{tdim}(X \cup Y)$. □

Note that by (iii) of Proposition 2.1.28, we conclude that if $X \subseteq M^n$ is definable, then $\text{tdim}(X) = \max\{\text{tdim}(C_i) : X = C_1 \cup ... \cup C_k \text{ is a cell decomposition of } X\}$.

The following technical lemma is needed to prove the last results about topological dimension that we will use in the rest of this work.

Proposition 2.1.29. *Let $\mathcal{M}$ be o-minimal and $S \subseteq M^m \times M^n$ be definable set. For a given $a \in M^m$, let $S_a = \{x \in M^n : (a, x) \in S\}$, and for each $d \in \{-\infty, 0, 1, \ldots, n\}$, let*

$$S(d) := \{a \in M^m : \textit{tdim}(S_a) = d\}$$

Then, for any $d \in \{0, 1, \ldots, n\}$, $S(d)$ is definable, and

$$\textit{tdim}\left(\bigcup_{a \in S(d)} \{a\} \times S_a\right) = \textit{tdim}(S(d)) + d$$

Proof. Let $\mathcal{D}$ be a cell decomposition of $M^m \times M^n$ that partitions S. Let $\pi : M^m \times M^n \to M^m$ denote the projection onto the first m coordinates, and note that given an $(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n})$-cell $C \in \mathcal{D}$, $\pi(C) \subseteq M^m$ is an $(i_1, \ldots, i_m)$-cell.

Claim: For each $a \in \pi(C)$, $C_a = \{x \in M^n : (a, x) \in C\}$ is an $(i_{m+1}, \ldots, i_{m+n})$-cell in M^n.

To prove this claim we will use induction. Let $n = 1$. If C is an $(i_1, \ldots, i_m, 0)$-cell, i.e. $C = \Gamma(f)$ for some $f \in C(X)$ and X an $(i_1, \ldots, i_m)$-cell, then $C_a = \{ f(a) \}$ which is an 0-cell. On the other hand, if C is an $(i_1, \ldots, i_m, 1)$-cell then there exists an $(i_1, \ldots, i_m)$-cell X and $f, g \in C(X)$ with $f < g$ such that $C = (f, g)$. Then C_a is the interval $(f(a), g(a))$ which is an 1-cell.

Suppose now that the claim is true for a certain n and let $C \subseteq M^m \times M^{n+1}$, and let $\pi_1 : M^m \times M^{n+1} \to M^m \times M^n$ be the projection onto the first $m + n$ coordinates, and set $D = \pi_1(C)$ which, by induction is an $(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n})$-cell . If C is an $(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n}, 0)$-cell, i.e. there exists $f \in C(D)$ such that $C = \Gamma(f)$, then note that $C_a = \Gamma(f_a)$, where $f_a : D_a \to M$ is given by $f_a(x) = f(a, x)$, and therefore C_a is an $(i_{m+1}, \ldots, i_{m+n}, 0)$-cell. On the other hand, if C is an $(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n}, 1)$-cell, then there exists $f, g \in C(D)$ with $f < g$ such that $C = (f, g)_D$. Again, by taking, $f_a, g_a : D_a \to M$ given by $f_a(x) = f(a, x)$ and $g_a(x) = g(a, x)$, we conclude that $C_a = (f_a, g_a)_{D_a}$ and therefore C_a is an $(i_{m+1}, \ldots, i_{m+n}, 1)$-cell.

To recap, we have a decomposition $\mathcal{D}$ of $M^n \times M^n$ that partitions S and we have established that, for any $(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n})$-cell C in $\mathcal{D}$, $\pi(C) \subseteq M^m$ is an $(i_1, \ldots, i_m)$-cell and $C_a \subseteq M^n$ is an $(i_{m+1}, \ldots, i_{m+n})$-cell, for each $a \in \pi(C)$. In particular, note that for each $a \in \pi(C)$,

$$\text{tdim}(C) = \text{tdim}(\pi(C)) + \text{tdim}(C_a)$$

Let $\pi(D)$ denote the set $\{\pi(C) : C \in \mathcal{D}\}$, and let $A \in \pi(\mathcal{D})$ be any cell. Let $C_1, \ldots, C_k$ be all the cells in $\mathcal{D}$ contained in S such that $\pi(C_i) = A$, for all $i = 1, \ldots, k$ and note that for each $a \in A$, $S_a = (C_1)_a \cup \ldots \cup (C_k)_a$ is a decomposition of S_a into finitely many cells.

Then

$$\begin{aligned}
\text{tdim}(S_a) &= \max_{i=1,\dots,k} \text{tdim}((C_i)_a) \\
&= \max_{i=1,\dots,k} \Big(\text{tdim}(C_i) - \text{tdim}(\pi(C_i))\Big) \\
&= \max_{i=1,\dots,k} \Big(\text{tdim}(C_i) - \text{tdim}(A)\Big) \\
&= \max_{i=1,\dots,k} \Big(\text{tdim}(C_i)\Big) - \text{tdim}(A)
\end{aligned}$$

Let $d = \text{tdim}(S_a)$, and note that, no matter which $a \in A$ we chose, we will allays have that $\text{tdim}(S_a) = d$, which means that $A \subseteq S(d)$. This implies that $S(d)$ is a finite union of cells: If $a \in S(d)$, then, because $\pi(\mathcal{D})$ is a decomposition of M^m, there exists a cell $A \in \pi(\mathcal{D})$ such that $a \in A$. As we just proved, this implies that $A \subseteq S(d)$, and as there are only finitely many cells by definition of a decomposition, $S(d)$ is the union of finitely many of cells, and thus definable.

Continuing where we left off, with $d = \text{tdim}(S_a)$, note the following

$$\begin{aligned}
d &= \max_{i=1,\dots,k} \Big(\text{tdim}(C_i)\Big) - \text{tdim}(A) \\
&= \text{tdim}\left(\bigcup_{i=1}^{k} C_i\right) - \text{tdim}(A) \qquad\qquad (*^1)
\end{aligned}$$

Now, for each $a \in A$, we have that $S_a = \bigcup_{i=1}^{k}(C_i)_a$. Furthermore, for each $i = 1, \dots, k$, note that $C_i = \bigcup_{a\in A}(\{\, a \,\} \times (C_i)_a)$. With this, we can deduce that:

$$\bigcup_{a\in A} \{\, a \,\} \times S_a = \bigcup_{a\in A} \bigcup_{i=1}^{k} \{\, a \,\} \times (C_i)_a = \bigcup_{i=1}^{k} C_i$$

Imputing this equality back on $(*^1)$, we obtain:

$$d = \text{tdim}\left(\bigcup_{a\in A} \{\, a \,\} \times S_a\right) - \text{tdim}(A)$$

And thus, rearranging the expression, we conclude that $\text{tdim}\left(\bigcup_{a\in A} \{\, a \,\} \times S_a\right) = \text{tdim}(A) + d$, and denote this expression by $(*^2)$ so we can refer back to it.

As we saw in the begging of the proof, there exists $A_1, \dots, A_l \in \pi\mathcal{D}$ such that $S(d) = A_1 \cup \dots \cup A_l$ is a cell decomposition of $S(d)$. As such:

$$\begin{aligned}
\text{tdim}\left(\bigcup_{a\in S(d)}\{a\}\times S_a\right) &= \text{tdim}\left(\bigcup_{i=1}^{l}\bigcup_{a\in A_i}\{a\}\times S_a\right)\\
&= \max_{i=1,\dots,l}\text{tdim}\left(\bigcup_{a\in A_i}\{a\}\times S_a\right)\\
&= \max_{i=1,\dots,l}(\text{tdim}(A_i)+d) && (\text{by } (*^2))\\
&= \text{tdim}\left(\bigcup_{i=1}^{l}A_i\right)+d\\
&= \text{tdim}(S(d))+d
\end{aligned}$$

And with this we conclude the proof for this proposition, as $d = 0, \dots, n$ is arbitrary. □

One of the main uses of this proposition is that it allows us to prove the following results, which are all properties that any ”good” definition of dimension should satisfy.

Corollary 2.1.30. *Let $\mathcal{M}$ be o-minimal.*

i) Let $S \subseteq M^m \times M^n$ be definable. Then tdim$(S) = \max_{0\leq d\leq n}($tdim$(S(d)) + d)$. Furthermore, tdim$(S) \geq$ tdim$(\pi(S))$, where $\pi : M^m \times M^n \to M^m$ is the projection onto the first m-coordinates;

ii) If $X \subseteq M^n$ is definable, and $f : X \to M^m$ is a definable map. Then, for each $d \in \{0, \dots, n\}$, the set $S_f(d) = \{a \in M^m :$ tdim$(f^{-1}(\{a\})) = d\}$ is definable and tdim$(f^{-1}(S_f(d))) =$ tdim$(S_f(d)) + d$. Furthermore tdim$(X) \geq$ tdim$(f(X))$;

iii) If $X \subseteq M^n$ and $Y \subseteq M^m$ are definable, then tdim$(X \times Y) =$ tdim$(X) +$ tdim(Y);

Proof. (i) Note that

$$S = \bigcup_{d=0}^{n}\bigcup_{a\in S(d)}\{a\}\times S_a$$

From this we can conclude that:

$$\begin{aligned}
\text{tdim}(S) &= \text{tdim}\left(\bigcup_{i=0}^{n} \bigcup_{a\in S(d)} \{a\} \times S_a\right) \\
&= \max_{d=0,\dots,n} \text{tdim}\left(\bigcup_{a\in S(d)} \{a\} \times S_a\right) && (\text{by } (iii) \text{ in Proposition 2.1.28}) \\
&= \max_{d=0,\dots,n} \big(\text{tdim}(S(d)) + d\big) && (\text{by Proposition 2.1.29})
\end{aligned}$$

Additionally, as $\pi(S) = \bigcup_{0\leq d\leq n} S(d)$, from Proposition 2.1.28, we conclude that tdim$(S) \geq$ tdim$(\pi(S))$.

(ii) Let $S = \{(f(x), x) : x \in X\} \subseteq M^m \times M^n$. Note that, for any $a \in M^m$,

$$S_a = \{x \in M^m : (a, x) \in S\} = \{x \in M^m : f(x) = a\} = f^{-1}(\{a\})$$

and thus, for any $d \in \{0, \dots, n\}$,

$$S(d) = \{a \in M^m : \text{tdim}(S_a) = d\} = \{a \in M^m : \text{tdim}(f^{-1}(\{a\})) = d\} = S_f(d)$$

Therefore, by Proposition 2.1.29, $S_f(d)$ is definable.

Now, note that tdim(X) = tdim(S) by Proposition 2.1.28, as the map $x \mapsto (f(x), x)$ is a definable bijection. By (i),

$$\text{tdim}(X) = \text{tdim}(S) \geq \text{tdim}(\pi(S)) = \text{tdim}(f(X))$$

(iii) Let $S = A \times B$. For each $a \in A$, $S_a = B$. This implies that $S(d) = \emptyset$ if $d \neq$ tdim(B) and $S(d) = A$ if $d =$ tdim(B). Recall that by definition, tdim$(\emptyset) = -\infty$. This together with (i) allows us to conclude the following:

$$\begin{aligned}
\dim(A \times B) &= \max_{0\leq d\leq n} (\text{tdim}(S(d)) + d) \\
&= \max\{-\infty, \text{tdim}(S(\text{tdim}(B)) + \text{tdim}(B)\} \\
&= \text{tdim}(A) + \text{tdim}(B)
\end{aligned}$$

□

When I stated Proposition 2.1.25, I mentioned that we would return to it later with a ”different approach.” That approach is, in fact, the concept of topological dimension. As it turns out, we have already established the very same properties listed in Proposition 2.1.25, but in the context of topological dimension. The next theorem shows that these results directly imply Proposition 2.1.25.

Theorem 2.1.31. *Let $\mathcal{M}$ be o-minimal and $X \subseteq M$ be definable. Then*

$$\textit{tdim}(X) = \dim(X)$$

Proof. Note that if $\mathcal{M}$ is not sufficiently saturated, we may consider a sufficiently saturated elementary extension of $\mathcal{M}$, so without loss of generality, assume that $\mathcal{M}$ is sufficiently saturated.

Claim: If $C \subseteq M^n$ is a cell, then tdim$(C) = \dim(C)$.

We will use induction over n.

Let $n = 1$, $C \subseteq M$ be a cell and, by adding any parameters as constants in the language, assume that C is $\emptyset$-definable. If C is an 0-cell, i.e. $C = \{x\}$, then $\dim(x/\emptyset) = 0$ and therefore $\dim(C) = 0$. Assume now that C is an 1-cell, and for the sake of contradiction assume that $\dim(C) = 0$. Then, for all $x \in C$, $\dim(x/\emptyset) = 0$, i.e. $\{x\}$ is $\emptyset$-definable. As $C \subseteq \bigcup_{x \in X}\{x\}$, by Lemma 1.8.7, C is finite, which contradicts the fact that tdim$(C) = 1$.

Assume now that the claim is true for n, let $C \subseteq M^{n+1}$ be a cell, and without loss of generality, assume again that C is $\emptyset$-definable. Start by assuming that C is an $(i_1, \dots, i_n, 0)$-cell, i.e. fixing $X = \pi(C) \subseteq M^n$, where π is the projection onto the first n coordinates, there exists $f \in C(X)$ such that $C = \Gamma(f)$. Let $\bar{a} = (a_1, \dots, a_n, a_{n+1}) \in C$, then $a_{n+1} = f(a_1, \dots, a_n)$, i.e. $a_{n+1} \in$ acl$(\bar{a} \setminus \{a_{n-1}\})$. This implies that $\dim(\bar{a}/\emptyset) = \dim((a_1, \dots, a_n)/\emptyset)$, i.e. $\dim(C) = \dim(X)$. By the induction hypothesis, $\dim(X) =$ tdim(X) and by de definition of topological dimension, tdim$(X) =$ tdim(C).

On the other hand, assume that C is an $(i_1, \dots, i_n, 1)$-cell. Then by setting $X = \pi(C) \subseteq M^n$, where π is the projection onto the first n coordinates, there exists $f, g \in C(X)$ with $f < g$ such that $C = (f, g)_X$. Let $\bar{a} = (a_1, \dots, a_n) \in X^n$ be a tuple such that $\dim(\bar{a}/\emptyset) = \dim(X)$, and let $(b_1, \dots, b_k)$ be a maximal independent sub-tuple of $\bar{a}$ (i.e. $k = \dim(X)$). I claim that there exists $x \in C_{\bar{a}}$ such that $(a_1, \dots, a_k, x)$ is independent. For that, assume that this is not the case and let $x \in C_{\bar{a}}$. Then, by our assumption, $(a_1, \dots, a_k, x)$ is not independent, which means that at least one of the following is true:

$$x \in \text{acl}(\bar{a}) \text{ or } a_i \in \text{acl}(\bar{a} \setminus \{a_i\} \cup \{x\}), \text{ for some } i = 1, \dots, k.$$

Regarding the latter, note that as $a_i \notin \mathrm{acl}(\bar{a} \setminus \{a_i\})$, due to the fact that $\bar{a}$ is independent, then we have that

$$a_i \in \mathrm{acl}(\bar{a} \setminus \{a_i\} \cup \{x\}) \setminus \mathrm{acl}(\bar{a} \setminus \{a_i\})$$

which, by the exchange property, implies that $x \in \mathrm{acl}(\bar{a})$, so wither way, we conclude that $x \in \mathrm{acl}(\bar{a})$.

As, by Proposition 2.0.2. in an ordered structure acl $=$ dcl, we conclude that each singleton $\{x\}$ in $C_{\bar{a}}$ is $\bar{a}$-definable, and by Lemma 1.8.7, this would imply that $C_{\bar{a}}$ is finite, contradicting the fact that $\mathrm{tdim}(C_{\bar{a}}) = 1$. Therefore, there exists $x \in C_{\bar{a}}$ such that $(a_1, \ldots, a_k, x)$ is independent.

This proves that $\dim(C) \geq \dim(X) + 1$. On the other hand, note that $\dim(C) > \dim(X) + 1$ is impossible. This is because, given $(a_1, \ldots, a_n, b) \in C$, any sub-tuple with length at least $\dim(X) + 2$ would contain at least $\dim(X) + 1$ many elements from $\{a_1, \ldots, a_n\}$, therefore such sub-tuple would not be independent as that would contradict the fact that $\dim((a_1, \ldots, a_n)/\emptyset) \leq \dim(X)$.

So, the fact that $\dim(C) = \dim(X) + 1$, in addition to the fact that, by the induction hypothesis, $\mathrm{tdim}(X) = \dim(X)$ allows us to conclude that:

$$\dim(C) = \dim(X) + 1 = \mathrm{tdim}(X) + 1 = \mathrm{tdim}(C)$$

Thus proving the claim.

Now for the general case, let $X \subseteq M^n$ definable set (again, we assume that X is $\emptyset$-definable) and let $X = C_1 \cup \ldots \cup C_k$ be a cell decomposition of X.

Claim: $\dim(X) = \max_{i=1\ldots,k} \dim(C_i)$.

Start by noting that for each $i = 1, \ldots, k$, $C_i \subseteq X$ implies that $\dim(C_i) \leq \dim(X)$, meaning that $\max_{i=1\ldots,k} \dim(C_i) \leq \dim(X)$. On the other hand, let $x \in X$ such that $\dim(x/\emptyset)$ is maximum among other elements of X. Then $x \in C_i$ for some $i = 1, \ldots, k$, which in particular, means that $\dim(X) \leq \dim(C_i)$, and therefore $\dim(X) \leq \max_{i=1\ldots,k} \dim(C_i)$.

With this, we can conclude that

$$\begin{aligned}
\dim(X) &= \max_{i=1\ldots,k} \dim(C_i) \\
&= \max_{i=1\ldots,k} \mathrm{tdim}(C_i) \\
&= \mathrm{tdim}(X)
\end{aligned}$$

Concluding the proof. $\square$

Due to this theorem, from now on, we will simply write $\dim(X)$ for both the geometric dimension and the topological dimension of X. Before I end this section I would like to present two results about dimension that will be useful later on.

Proposition 2.1.32. *Let $X \subseteq M^n$ be definable. Then $\dim X \geq k+1$ if and only if there exists a definable equivalence relation E on X such that E has an infinite number of equivalence classes of dimension at least k.*

Proof. For the proof see Proposition 1.8 of [9] $\square$

Lemma 2.1.33. *Let $\mathcal{M}$ be o-minimal, let $S \subseteq M^n$ be definable and let E be a definable equivalence relation on S such that:*

(i) There exists a definable set $Y \subseteq M^m$ and a definable surjective map $f : S \to Y$ such that xEy if and only if $f(x) = f(y)$;

(ii) each equivalence class of E has dimension k.

Then $\dim(S/E) = \dim(S) - k$, where S/E is identified with Y (see Section 1.9).

Proof. The proof is a simple application of Corollary 2.1.30. For $d = 0, \ldots, n$, define

$$S_f(d) = \{y \in Y : \dim f^{-1}(\{y\}) = d\}$$

Note that if $y \in Y$, then $f^{-1}(\{y\})$ is simply an equivalence class of E and therefore, $\dim f^{-1}(\{y\}) = k$ for all $y \in Y$. In particular, $S_f(k) = Y$. From (ii) in Proposition 2.1.30, we have that $\dim(f^{-1}(S_f(k))) = \dim(S_f(k)) + k$, which simplifies to $\dim(S) = \dim(Y) + k$, and rearranging this expression we obtain, $\dim(Y) = \dim(S) - k$. $\square$

In particular, if $\mathcal{M}$ has elimination of imaginaries, condition (i) is always verified.

2.2. Definable Groups in O-minimal Structures

Before starting I would like to fix the following:

> For the rest of this chapter, unless stated otherwise, all structures we are working with are assumed to be sufficiently saturated.

This assumption is mainly to simplify the proofs, and in general the results we will see remain true even if $\mathcal{M}$ is not saturated, and can be proven by passing to a sufficiently saturated elementary extension. Additionally, until now, we have used the notation $A\bar{a}$ to denote the set $A \cup \bar{a}$, but from this point on we will denote this union by $A^\frown\bar{a}$ to avoid confusion.

The principal object of study in this section are definable groups. A definable group G in some o-minimal structure $\mathcal{M}$ is a definable subset $G \subseteq M^n$, for some $n \geq 1$, with a definable group operation, i.e. a group operation that is definable as a function from $G \times G$ to G.

We start by proving some technical lemmas before presenting the first main theorem of this section.

Lemma 2.2.1. *Let $f : M^n \to M^n$ be an A-definable bijective map, and $\bar{a} \in M^n$. Then $\dim(\bar{a}/A) = \dim(f(\bar{a})/A)$.*

Proof. Add the elements of A to the language so that we may assume that $A = \emptyset$. Start by noting that $(\bar{a}, f(\bar{a})) \subseteq$ acl$(\bar{a})$ and $(\bar{a}, f(\bar{a})) \subseteq$ acl$(f(\bar{a}))$. Let $\bar{a}' \subseteq \bar{a}$ be a maximal independent sub-tuple of $\bar{a}$.

Claim: $\bar{a}'$ is a maximal independent sub-tuple of $(\bar{a}, f(\bar{a}))$

Trivially, $\bar{a}'$ is an independent sub-tuple of $(\bar{a}, f(\bar{a}))$. Assume, aiming for a contradiction, that $\bar{a}'$ is not a maximal independent sub-tuple, i.e. there exists $b \in (\bar{a}, f(\bar{a})) \setminus \bar{a}'$ such that $\bar{a}'^\frown b$ is independent. Start by noting that b can not be an element of $\bar{a}$, since that would contradict the maximality of $\bar{a}'$ inside of $\bar{a}$. So b has to be an element of $f(\bar{a})$. However, we had already established that $f(\bar{a}) \subseteq$ acl$(\bar{a})$, and as $\bar{a}'$ is a maximal independent sub-tuple of $\bar{a}$, we have that $\bar{a} \subseteq$ acl$(\bar{a}')$, i.e. $f(\bar{a}) \subseteq$ acl$(\bar{a}')$, and in particular, $b \in$ acl$(\bar{a}')$, which contradicts the fact that $\bar{a}'b$ is independent.

This claim proves that $\dim((\bar{a}, f(\bar{a}))/\emptyset) = \dim(\bar{a}/\emptyset)$. An analogous argument shows that $\dim((\bar{a}, f(\bar{a}))/\emptyset) = \dim(f(\bar{a})/\emptyset)$ and therefore $\dim(\bar{a}/\emptyset) = \dim(f(\bar{a})/\emptyset)$. $\square$

Recall the definition of a generic point from Definition 2.1.23.

Lemma 2.2.2. *Let G be a definable group in an o-minimal structure $\mathcal{M}$.*

i) Let $b \in G$ and let a be a generic of G over b. Then $b \cdot a$ is a generic of G over b;

ii) For any $b \in G$, there exists generics b_1, b_2 of G over b such that $b = b_1 \cdot b_2$.

Proof. i) We have that $\dim(G) = \dim(a/b)$, and by Lemma 2.2.1, $\dim(b \cdot a/b) = \dim(a/b)$, meaning that $\dim(G) = \dim(b \cdot a/b)$, i.e. $b \cdot a$ is a generic of G over b.

ii) Let a be a generic of G over b. This implies, by Lemma 2.2.1 that a^{-1} is generic over b, and by (i), $b \cdot a^{-1}$ is also generic over b. Letting $b_1 = b \cdot a^{-1}$ and $b_2 = a$ we get the desired result. □

Lemma 2.2.3. *Let G be a definable group in an o-minimal structure and let H be a definable subgroup of G. Then $\dim H = \dim G$ if and only if H has finite index in G.*

Proof. Start by noting that for all $a \in G$, $\dim(H) = \dim(Ha)$, as left multiplication is a definable bijection.

Assume that $\dim H = \dim G$ and consider the definable equivalence relation E on G given by $xEy \Leftrightarrow xy^{-1} \in H$. The equivalence classes of E are $\{Ha : a \in G\}$ and therefore, by Proposition 2.1.32 if there were infinitely many equivalence classes of E, we would have that $\dim G \geq \dim H + 1$ which is not true. So H has finite index on G.

On the other hand, assume now that H has finite index on G, i.e. there exists $a_1, \ldots, a_k \in G$ such that

$$G = Ha_1 \cup \ldots \cup Ha_k$$

by Proposition 2.1.28 (iii), we get that $\dim G = \dim H$, as $\dim(Ha_i) = \dim(H)$ for all $i = 1, \ldots, k$. □

I now introduce the concept of large sets, which will play a crucial role in the first main theorem of this section. Intuitively, given a definable set X and a definable subset $Y \subseteq X$, we say that Y is large in X if it has the same dimension as X, and removing Y from X leaves behind something of strictly smaller dimension.

Definition 2.2.4. Let $\mathcal{M}$ be o-minimal, $X \subseteq M^n$ be a definable set and $Y \subseteq X$ be a definable subset. We say that Y is *large* in X or that Y is a *large subset* of X, if $\dim(X \setminus Y) < \dim(X)$.

The following is an alternative characterization of large sets, which is sometimes useful.

Lemma 2.2.5. *Let $\mathcal{M}$ be o-minimal and $Y \subseteq X$ be A-definable sets. Then Y is large in X if and only if every generic point a of X over A is an element of Y.*

Proof. Fix $\dim X = n$.

Start by assuming that Y is large in X and $a \in X$ is a generic over A. If $a \notin Y$, then $a \in X \setminus Y$, which is an A-definable set. By the definition of dimension, this implies that $\dim(X \setminus Y) \geq \dim(a/A) = n$, which contradicts the fact that Y is large in X.

Assume now that any generic of X over A is in Y. If Y is not large in X, i.e. $\dim(X \setminus Y) = n$, then there exists $w \in X \setminus Y$ such that $\dim(w/A) = n$. However, this would make w a generic of X over A and therefore an element of Y, which is a contradiction. So Y is large in X. $\square$

Before proceeding further, we need the following technical model-theoretic lemma:

Lemma 2.2.6. *Let $\mathcal{M}$ be a structure (not necessarily saturated) and $A \subseteq B \subseteq M$ be subsets. If $p \in S_n^{\mathcal{M}}(A)$ is finitely satisfiable in A, then there exists a type $q \in S_n^{\mathcal{M}}(B)$ such that $p \subseteq q$ and q is finitely satisfiable in A.*

Proof. Let $p \in S_n^{\mathcal{M}}(A)$ be finitely satisfiable over A, and consider the set

$$\Sigma = p \cup \{\neg\phi : \phi \in \mathcal{L}_B \text{ and } \mathcal{M} \not\models \phi[a], \text{ for all } a \in A\}$$

I start by showing that Σ is a (incomplete) type over B, i.e. that $\text{Th}_B(\mathcal{M}) \cup \Sigma$ is satisfiable. Let $p_0 \cup \{\neg\phi_1, \dots, \neg\phi_k\}$ be a finite subset of Σ with $p_0 \subseteq p$. As p is finitely satisfiable over A, there exists $a \in A$ such that $\mathcal{M} \models \psi[a]$, for all $\psi \in p_0$. By construction, $\mathcal{M} \not\models \phi_i[a]$, for all $i = 1, \dots, k$, i.e. $\mathcal{M} \models \neg\phi_i[a]$ for all $i = 1, \dots, k$. Thus $\text{Th}_B(\mathcal{M}) \cup \Sigma$ is finitely satisfiable, and by compactness, satisfiable. By Lemma 1.7.8, there exists $q \in S_n^{\mathcal{M}}(B)$ such that $p \subseteq \Sigma \subseteq q$. All that is left to show is that q is finitely satisfiable over A. Let $\Lambda \subseteq q$ be a finite subset, and assume, for the sake of contradiction, that for all $a \in A$, we have $\mathcal{M} \not\models \bigwedge_{\phi \in \Lambda} \phi[a]$. Then, by construction, $\neg \bigwedge_{\phi\in\Lambda} \phi \in \Sigma \subseteq q$. This would mean that $\neg \bigwedge_{\phi\in\Lambda} \phi \in q$, which is a contradiction since $\Lambda \subseteq q$ implies that $\bigwedge_{\phi\in\Lambda} \phi \in q$. $\square$

Lemma 2.2.7. *Let G be a definable group in an o-minimal structure $\mathcal{M}$ and let X be a large definable subset of G. Then finitely many translations of X cover G.*

Proof. Let $\mathcal{M}_0 \preceq \mathcal{M}$ be a small elementary substructure such that X is M_0-definable. The following claim is at the heart of this lemma:

Claim: For each $a \in G$ there exists $b \in G(\mathcal{M}_0)$ such that $a \in bX$.

Before proceeding, recall that $G(\mathcal{M}_0)$ denotes the elements of M_0 that satisfy the formula that defines G. In particular,note that $G(\mathcal{M}_0) \neq \emptyset$ as $\mathcal{M} \models \exists x \phi(x)$ and $\mathcal{M}_0 \equiv \mathcal{M}$, where ϕ is a formula with parameters from M_0 that defines G. With this in mind, fix $a \in G$.

Let c^* be a generic of G over M_0 and consider the type $p = \text{tp}^{\mathcal{M}}(c^*/M_0)$. Then p is finitely satisfiable in M_0, and by Lemma 2.2.6, there exists a type $q \in S_n^{\mathcal{M}}(M_0 \cup a)$ that is finitely satisfiable in M_0. As $\mathcal{M}$ is sufficiently saturated, there exists $c \in \mathcal{M}$ such that $q = \text{tp}^{\mathcal{M}}(c/M_0 \cup a)$.

Note that:

a) $c \in G$, as $\text{tp}^{\mathcal{M}}(c^*/M_0) = p = \text{tp}(c/M_0)$;

b) c is a generic of G over M_0. This is because c also realizes p, meaning that

$$\dim(c/M_0) = \dim(p) = \dim(c^*/M_0)$$

Additionally, this also implies that c is a generic of G over $M_0 \cup a$. To see this let c' be a maximal sub-tuple of c independent over M_0 and let a' be a maximal sub-tuple of a independent over M_0. I claim that $a' {}^\frown c'$ is an independent sub-tuple of $a {}^\frown c$ over M_0. To see this, assume that $a' {}^\frown c'$ is not independent over M_0, and let $d \in a' {}^\frown c'$ be an element such that $d \in \text{acl}(M_0 \cup a' {}^\frown c' \setminus \{d\})$.

Write $c = (c_1, \ldots, c_n)$ and without loss of generality, assume that $c' = (c_1, \ldots, c_k)$.

We will start by examining the case where $d \in a'$. Let $\phi(v, w_1, \ldots, w_k)$ be an $\mathcal{L}_{M_0 \cup a' \setminus \{d\}}$-formula such that $\{d\} = \phi(\mathcal{M}, c')$. Consider the $\mathcal{L}_{M_0 \cup a}$-formula $\psi(w_1, \ldots, w_n)$ given by:

$$\forall v \left(\phi(v, w_1, \ldots, w_k) \leftrightarrow v = d \right)$$

Then, by definition of ϕ, we have $\mathcal{M} \models \phi[c]$, i.e. $\phi \in \text{tp}^{\mathcal{M}}(c/M_0 \cup a)$. However, by definition of c, the type $\text{tp}^{\mathcal{M}}(c/M_0 \cup a)$ is finitely satisfiable over M_0, i.e. there exists $\xi = (\xi_1, \ldots, \xi_n) \in M_0^n$ such that $\mathcal{M} \models \psi[\xi]$, and in particular, by the way we built the formula ψ, this implies that $\{d\} = \phi(\mathcal{M}, \xi_1, \ldots, \xi_k)$. Because $\xi_1, \ldots, \xi_k$ are elements of M_0, we conclude that $d \in \text{acl}(M_0 \cup a' \setminus \{d\})$, which contradicts the fact that a' is independent, so the case where $d \in a'$ is impossible.

On the other hand, assume that $d \in c'$. Let $a'' \subseteq a'$ be minimal such that $d \in \text{acl}(M_0 \cup a'' {}^\frown c' \setminus \{d\})$. In particular, note that $a'' \neq \emptyset$, since otherwise, we would get that $d \in \text{acl}(M_0 \cup c' \setminus \{d\})$, which would contradict the fact that c' is independent over M_0. Let

$\xi \in a''$ be any element, and note that by minimality of a'', we have that $d \notin \text{acl}(M_0 \cup a''{}^\frown c' \setminus \{d, \xi\})$, i.e.:

$$d \in \text{acl}(M_0 \cup a''{}^\frown c' \setminus \{d\}) \setminus \text{acl}(M_0 \cup a''{}^\frown c' \setminus \{d, \xi\})$$

By the exchange property:

$$\xi \in \text{acl}(M_0 \cup a''{}^\frown c' \setminus \{\xi\})$$

But then, we would have that $\xi \in \text{acl}(M_0 \cup a'{}^\frown c' \setminus \{\xi\})$, and because $\xi \in a'' \subseteq a'$, this would mean that ξ falls into the previous case, which we have already seen is impossible

To recap, we just proved that if c' is a maximal sub-tuple of c independent over M_0 and if a' is a maximal sub-tuple of a independent over M_0, then $a'{}^\frown c'$ is an independent sub-tuple of $a^\frown c$ over M_0. Note that this implies that $\dim(a^\frown c/M_0) \geq \dim(a/M_0) + \dim(c/M_0)$. On the other hand, if $a' \subseteq a$ and $c' \subseteq c$ are sub-tuples such that $a'{}^\frown c'$ is a maximal independent sub-tuple of $a^\frown c$ over M_0, then a' is an independent sub-tuple of a over M_0 and c' is an independent sub-tuple of c over M_0, meaning that $\dim(a^\frown c/M_0) \leq \dim(a/M_0) + \dim(c/M_0)$, and thus establishing that $\dim(a^\frown c/M_0) = \dim(a/M_0) + \dim(c/M_0)$.

From Lemma 2.1.27, we also know that $\dim(a^\frown c/M_0) = \dim(c/M_0 \cup a) + \dim(a/M_0)$, and so we get that $\dim(c/M_0 \cup a) = \dim(c/M_0)$, meaning that c is a generic of G over $M_0 \cup a$.

Now, as X is large in G, then $X \cdot a^{-1}$ is also large in G by Proposition 2.1.28. By Lemma 2.2.5, we have that $c \in X \cdot a^{-1}$, i.e. $a \in c^{-1}X$. Let $\phi(\bar{v})$ be the formula that defines G and $\psi(\bar{v})$ be the formula that defines X. Consider the formula

$$\chi(\bar{v}) = \phi(\bar{v}) \wedge \exists \bar{x}(\psi(\bar{x}) \wedge a = \bar{v}^{-1} \cdot \bar{x})$$

As $\mathcal{M} \models \chi[c]$, we conclude that $\chi \in \text{tp}^{\mathcal{M}}(c/M_0 \cup a)$. Moreover, this type is finitely satisfiable in M_0, meaning that there exists $b \in M_0$ such that $\mathcal{M} \models \chi[b]$. In particular $\mathcal{M} \models \phi[b]$ (i.e. $b \in G(\mathcal{M}_0)$) and $a \in b^{-1}X$, which proves the claim.

The set G is definable and $\Sigma := \{bV, b \in G(\mathcal{M}_0)\}$ is a family of definable sets that cover G (by the claim). By Lemma 1.8.7, there exists $b_1, \ldots, b_k$ such that $G = b_1V \cup \ldots \cup b_kV$. $\square$

2.2.1 The t-Topology

The following is a central theorem in the development of the theory of definable groups in o-minimal structures. As we will see, it is of paramount importance and enables us to establish strong results about the nature of such groups, so much so that, from this point onward, we will use it frequently - either directly or indirectly through results whose proofs depend on it.

Theorem 2.2.8. *Let G be an A-definable group in an o-minimal structure $\mathcal{M}$ with $\dim G = n$. Then, there exists a large A-definable subset V of G and a unique topology t on G such that:*

i) Multiplication and inversion are continuous with respect to the topology t, i.e. G equipped with the topology t is a topological group;

ii) V is a finite disjoint union of A-definable sets $U_1, \ldots, U_k$ such that each U_i is open with respect to t and there is an A-definable homeomorphism between U_i (with the topology induced by t) and an open subset of M^n.

When talking about topological concepts with respect to the t-topology it is common to write t- before the property to emphasize which topology we are working with. For example, if $A \subseteq G$ is a subset of G, then A being open means that it is open with respect to the topology of M^n, while A being t-open means that it is open with respect to the t topology on G. The same goes for other topological concepts, like t-closed, t-connected, t-compact, etc...

I will only sketch the construction of this topology, highlighting the key aspects that we will need in order to prove certain properties about it. For the full construction and detailed proof, see [9].

In a nutshell, the construction of the topology goes as follows: The author constructs a large definable open set $V \subseteq G$ and defines $X \subseteq G$ to be t-open if, for all $g \in G$, the set $gX \cap V$ is open. In particular, V is t-open. Furthermore, the topology induced in V by t coincides with the topology in M^n, i.e. $X \subseteq V$ is t-open if and only if it is open in the ambient topology.

The basis for this topology is definable. In fact, given g a generic of G and $\{U_i : i \in I\}$ a definable neighborhood basis of g, then the family $\{hU_i : i \in I, h \in G\}$ forms a basis for the t topology in G.

The rest of this subsection is dedicated to exploring some results that follow from considering G with the t topology. These results have been included solely based on their relevance for later sections of Chapter 3, so by all means, they are not intended to be considered the main consequences or the most important corollaries of the existence of the t topology.

Lemma 2.2.9. *Any definable subset $Z \subseteq G$ is a finite union of t-locally closed definable subsets of G.*

Proof. Let $Z \subseteq G$ be definable and V be the large open subset of G used in the construction of the topology t. By Lemma 2.2.7, there exists $a_1, \ldots, a_k \in G$ such that $G = a_1V \cup \ldots \cup a_KV$ which means that we can write Z as

$$Z = (Z \cap a_1V) \cup \ldots \cup (Z \cap a_kV)$$

it is enough to show that for each $i = 1, \ldots, k$, the set $Z \cap a_iV$ is a finite union of t-locally closed definable subsets of G. Start by noting that $Z \cap a_iV = a_i(a_i^{-1}Z \cap V)$. The set $a_i^{-1}Z \cap V$ is a definable subset of V, and by o-minimality we can write $a_i^{-1}Z \cap V = C_1 \cup \ldots \cup C_l$ where each C_j is a cell. From Proposition 2.1.13, each C_j is locally closed and because the t-topology in V is equal to the topology in M^n, each C_j is t-locally closed. As left multiplication by a_i is an homomorphism, a_iC_j is t-locally closed for each j, meaning that

$$Z \cap a_iV = a_i(a_i^{-1}Z \cap V) = a_iC_1 \cup \ldots \cup a_iC_l$$

is the finite union of t-locally closed sets. $\square$

One consequence of the cell decomposition theorem, is any definable set $X \subseteq M^n$ can be written as a finite disjoint union of definably connected definable sets, with respect to the ambient topology of M^n. We will now see that this is also the case for definable subsets of G.

Lemma 2.2.10. *Any definable subset $Z \subseteq G$ is a finite disjoint union of definably t-connected definable sets.*

Proof. Let $Z \subseteq G$ be definable and V be the large open subset of G used in the construction of the topology t. By Lemma 2.2.7, there exists $a_1, \ldots, a_k \in G$ such that $G = a_1V \cup \ldots \cup a_KV$. Let $X_1 = Z \cap a_1V$ and for each $i = 2, \ldots, k$, define recursively

$$X_i = (Z \cap a_iV) \setminus (X_1 \cup \ldots \cup X_{i-1})$$

Then:

(i) $Z = X_1 \cup \ldots \cup X_k$;

(ii) The sets $X_1, \ldots, X_k$ are definable and pairwise disjoint;

(iii) For each $i = 1, \ldots, k$, $X_i \subseteq a_i V$.

it is enough to show that each X_i is a finite disjoint union of definably t-connected definable sets. We have that $a_i^{-1}X \subseteq V$ is a definable subset of V. By cell decomposition, we can write $a_i^{-1}X$ as a disjoint union $C_1 \cup \ldots \cup C_l$ of cells. From Proposition 2.1.15, each C_j is definably connected and thus definably t-connected, as the t-topology in V coincides with the topology of M^n. Left multiplication by a_i is a homeomorphism, meaning that each a_iC_j is definably t-connected. As $X_i = a_iC_1 \cup \ldots \cup a_iC_k$, we conclude the result. $\square$

In particular, the group G is itself a finite union of definably t-connected definable subsets.

Before continuing, I need two general lemmas about topological spaces and topological groups, which are pretty standard and as such, I will not present a proof.

Lemma 2.2.11. *Let X be a topological space and $Y \subseteq X$ a non-empty subset. Let $A \subseteq X$ be a boolean combination of open sets, and let $B = X \setminus A$. Then either $Y \cap A$ or $Y \cap B$ has non-empty interior in Y (with the induced topology).*

Lemma 2.2.12. *Let G be a topological group and H be a subgroup of G. Then:*

(i) The closure $\bar{H}$ of H is a subgroup of G;

(ii) If H has non-empty interior, then H is open;

(iii) If H is open, then H is closed.

With these two lemmas, we can prove the following interesting corollary:

Corollary 2.2.13. *Any definable subgroup $H \leq G$ is t-closed.*

Proof. Let H be a definable subgroup of G, and by Lemma 2.2.12, the closure with respect to t $\bar{H}$ is also a subgroup of G. It is enough to show that $\bar{H} = H$. We want to apply Lemma 2.2.11 with $X = Y = \bar{H}$ and $A = H$. For that, we first need to show that H is a boolean combination of open sets of $\bar{H}$, with the topology induced by t. By Lemma 2.2.9, we can write $H = F_1 \cup \ldots \cup F_k$ where each F_i is a t-locally closed definable set in G, and thus a boolean combination of t-open sets. Intersecting with $\bar{H}$ we get that

$H = (\bar{H} \cap F_1) \cup \ldots \cup (\bar{H} \cap F_k)$, and now each $\bar{H} \cap F_i$ is locally closed in $\bar{H}$ with respect to the topology induced by t, thus a boolean combination of open sets if $\bar{H}$.

Now, note that $\bar{H} \setminus H$ has empty interior in $\bar{H}$, and so by Lemma 2.2.11, H has non-empty interior in $\bar{H}$. By Lemma 2.2.12 H is open in $\bar{H}$ and thus, by the same lemma, closed in $\bar{H}$ and thus $H = \bar{H}$, meaning that H is closed in G. □

The following result is due to M. Edmundo [15], and heavily relies on the existence of the t-topology.

Theorem 2.2.14. *Let G be a definable group and H be a definable normal subgroup of G. Then the family $\{gH, g \in G\}$ has definable choice, i.e. G/H is definable.*

This theorem together with Lemma 2.1.33 allows us to conclude the following.

Corollary 2.2.15. *Let G be a definable group and H a definable subgroup. Then $\dim(G/H)$ is equal to $\dim G - \dim H$.*

We already know that any definable subgroup of a definable group is t-closed. The following lemma gives equivalent conditions for it to also be t-open.

Lemma 2.2.16. *Let $H \leq K \leq G$ be definable subgroups of G. The following are equivalent:*

(i) H is t-open in K (with the induced topology);

(ii) H has finite index on K;

(iii) $\dim H = \dim K$.

Proof. Note that $(ii) \Leftrightarrow (iii)$ by Lemma 2.2.3.

$(ii) \Rightarrow (i)$ Assume now that H has finite index on K. Then, there exists $a_1, \ldots, a_k \in K$ such that $K = H \cup a_1H \cup \ldots \cup a_kH$. By Corollary 2.2.13, H is t-closed and as left multiplication is an homeomorphism, each a_iH is t-close. As we can write H as $H = K \setminus \bigcup_{i=1}^{k} a_iH$, we conclude that H is t-open.

$(i) \Rightarrow (ii)$ Assume that H is t-open in K, which by Corollary 2.2.13, implies that H is actually t-clopen. By Lemma 2.2.10, we can write $K = K_1 \cup \ldots \cup K_m$, where each K_i is a definable, definably t-connected subset of G, and the union is disjoint. Now, note that for any $g \in K$, gH is t-clopen, and therefore, for each $i = 1, \ldots, m$, either $gH \cap K_i = \emptyset$ or $gH \cap K_i = K_i$, since any other option would contradict the fact that K_i is definably t-connected, as it would contain a proper clopen subset. This means that there are at most 2^m possibilities for gH and so the index of H in K is at most 2^m. $\square$

Theorem 2.2.17. *Let G be a definable group. Then there are only finitely many definable subgroups H of G such that $\dim H = \dim G$.*

Proof. Let H be a definable subgroup of G with $\dim H = \dim G$. Start by noting that H is t-clopen, by Corollary 2.2.13 and Lemma 2.2.16. Let V be the large definable subset of G used in Theorem 2.2.8 to define the t-topology. Decompose V into the disjoint union of k cells, and note that each of them is t-definably connected. As H is t-clopen, then each cell is either disjoint from H or contained in H. This leaves us with 2^k possible sets of the form $H \cap V$ where H is a definable subgroup of G with $\dim(G) = \dim(H)$. As $H = \bar{H} = \overline{H \cap V}$, we get that there are at most 2^k definable subgroups of G with $\dim(G) = \dim(H)$. $\square$

The following is an important and often very useful corollary of this theorem:

Corollary 2.2.18. *There is no infinite strictly descending chain of definable subgroups of G (this property is called the descending chain condition, sometimes abbreviated to DCC).*

Proof. Assume that there exists an infinite descending chain $G < H_1 < \ldots < H_k < \ldots$. Then $\dim G \leq \dim H_1 \leq \ldots \leq \dim H_k \leq \ldots$. As $\dim H_k$ is finite for all k, there exists $N \in \mathbb{N}$ such that $n > N$ implies that $\dim H_n = \dim H_N$. However, this would imply that there exists an infinite amount of subgroups of H_N with $\dim H_n = \dim H_N$, which contradicts Theorem 2.2.17. $\square$

One example of an application of the DCC in definable groups is the following corollary, which guarantees the existence of a smallest definable subgroup containing any given subset, even if said subset is not definable.

Corollary 2.2.19. *Let G be a definable group and $A \subseteq G$ be a subset. Then, there exists a definable subgroup $\langle A \rangle_{def}$ containing A such that, for any definable subgroup T with $A \subseteq T$, we have that $\langle A \rangle_{def} \subseteq T$.*

Proof. Let

$$\Gamma = \{T : T \text{ is a definable subgroup of } G \text{ with } A \subseteq T\}$$

I claim that $\langle A\rangle_{\text{def}}$ can be taken to be $\bigcap_{T\in\Gamma} T$. For this, it is enough to verify that $\bigcap_{T\in\Gamma} T$ is definable, as the minimality comes trivially.

Let $\kappa = |\Gamma|$ and let $\Gamma = \{T_\alpha : \alpha < \kappa\}$ be an enumeration of Γ. For each ordinal $\alpha < \kappa$, we define

$$K_\alpha = \begin{cases} \bigcap_{\beta\leq\alpha} T_\beta & \text{, if } \alpha \text{ is successor ordinal} \\ \bigcap_{\beta<\alpha} T_\beta & \text{, if } \alpha \text{ is a limit ordinal} \end{cases}$$

Start by noting that, for any $\alpha < \omega$, K_α is a finite intersection of definable sets and hence, definable. Also, note that $K_\omega = K_N$ for some $N < \omega$ by the DCC.

Claim: K_α is definable for all $\alpha < \kappa$.

We proceed via induction.

Let $\alpha^+ < \kappa$ be a successor ordinal, and assume that K_α is definable. Then $K_{\alpha^+} = K_\alpha \cap T_{\alpha^+}$ which is definable.

Now, let $\alpha < \kappa$ be a limit ordinal and assume that for all $\beta < \kappa$, K_β is definable.

Note that there exists $\beta < \alpha$ such that for all ordinals γ with, $\beta \leq \gamma < \alpha$ we have that $K_\beta = K_\gamma$. Indeed, if this is not the case, one could build an infinite countable descending chain of definable subgroups, which would contradict the DCC. This means that $K_\alpha = \bigcap_{\gamma<\alpha} T_\gamma = \bigcap_{\gamma<\alpha} K_\gamma = K_\beta$ which is definable.

The fact that K_α is definable for each ordinal $\alpha < \kappa$ means that $\{K_\alpha : \alpha < \kappa\}$ forms a descending chain of definable subgroups of G, with $K_\alpha \supseteq K_\beta$ when $\alpha \leq \beta < \kappa$. By the same argument we did before, there exists $\beta < \kappa$ such that for all $\beta \leq \gamma < \kappa$ implies that $K_\beta = K_\alpha$. As such $K_\beta = \bigcap_{\alpha<\kappa} K_\alpha = \bigcap_{\alpha<\kappa} T_\alpha = \bigcap_{T\in\Gamma} T = \langle A\rangle_{\text{def}}$. □

2.3. Definable spaces

The aim of this section is to isolate some important properties of the t-topology by defining a new object of interest: Definable spaces. Intuitively, a definable space is a set (not necessarily definable) such that each point has a ”definable-like” neighborhood. This section mainly follows Chapter 10 in [7].

Definition 2.3.1. Let X be an arbitrary set (not necessarily definable). A *definable atlas* on X relative to $\mathcal{M}$ is a finite family $(U_i, \phi_i)_{i\in I}$ such that:

i) For each $i \in I$, $U_i \subseteq X$ and $X = \bigcup_{i\in I} U_i$;

ii) For each $i \in I$, $\phi_i(U_i) \subseteq M^{n_i}$ is definable and $\phi_i : U_i \to \phi_i(U_i)$ is a bijection;

iii) For each $i, j \in I$, $\phi_i(U_i \cap U_j)$ is definable and open in $\phi_i(U_i)$ with respect to the topology induced by $\mathcal{M}$;

iv) For each $i, j \in I$, each map $\phi_{ji} := \phi_j \circ \phi_i^{-1} := \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable homeomorphism.

Given a definable atlas on a set X, there is a very natural topology we can build based on the following result from topology.

The following is a well known result from topology.

Theorem 2.3.2. *Let X be a set and $\{X_i\}_{i\in I}$ be a collection of subsets of X such that $X = \bigcup_{i\in I} X_i$. Suppose that each set X_i has a topology τ_i such that: for each $i, j \in I$, the set $X_i \cap X_j$ is open in X_i and X_j and the topologies on $X_i \cap X_j$ induced by X_i and X_j are the same. Then, there exists a unique topology on X that induces in each X_i the topology τ_i.*

As I mentioned earlier, we will use this theorem to define a topology on X from a definable atlas.

Corollary 2.3.3. *Let X be a set and $\{U_i\}_{i\in I}$ be a collection of subsets of X such that $X = \bigcup_{i\in I} U_i$. For each $i \in I$, let $\phi_i : U_i \to U_i'$ be a bijection from U_i to a topological space U_i' such that:*

1. *For each $i, j \in I$, $\phi_i(U_i \cap U_j)$ is open in U_i';*

2. *For each $i, j \in I$, each map $\phi_{ji} := \phi_j \circ \phi_i^{-1} := \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a homeomorphism.*

Then, there exists a unique topology on X such that each U_i is open and the maps $\phi : U_i \to U_i'$ are homeomorphisms. In particular, $Y \subseteq X$ is open if and only if $\phi_i(Y \cap U_i)$ is open for each $i \in I$.

Now, let X be a set and $(U_i, \phi_i)_{i\in I}$ a definable atlas. For each $i \in I$, the set $\phi_i(U_i)$ is a topological space when considering the topology induced by the order topology on the ambient space $\mathcal{M}$. By the definition of definable atlas, the cover $\{U_i\}_{i\in I}$ checks all the boxes needed to apply Corollary 2.3.3. Given a definable atlas $\Gamma = (U_i, \phi_i)_{i\in I}$, we denote by τ_Γ the topology induced by Γ in X defined this way.

Definition 2.3.4. Let X be a set and Γ and Λ be two definable atlas over $\mathcal{M}$. We say that Γ and Λ are *equivalent* if $\Gamma \cup \Lambda$ is a definable atlas.

Note that "being equivalent" is an equivalence relation between two definable atlases.

Lemma 2.3.5. *Let X be a set and Γ and Λ be two equivalent definable atlas over $\mathcal{M}$. Then Γ and Λ induce the same topology on X.*

Proof. Let $\Gamma = (U_i, \phi_i)_{i \in I}$ and $\Lambda = (V_j, \psi_j)_{j \in J}$. For each $i \in I$, U_i is open with respect to $\tau_{\Gamma \cup \Lambda}$ and ϕ_i is an homeomorphism. By uniqueness, $\tau_\Gamma = \tau_{\Gamma \cup \Lambda}$. An analogous argument shows that $\tau_\Lambda = \tau_{\Gamma \cup \Lambda}$. □

With this, we have the following definition of definable space.

Definition 2.3.6. A *definable space* X relative to $\mathcal{M}$ is a set X together with an equivalence class of definable atlases on X. If Γ is a definable atlas from this equivalence class, we say that Γ is compatible with X.

Equivalently, because of Corollary 2.3.3 and Lemma 2.3.5, we can define a definable space X as a topological space with the unique topology induced by its equivalence class of atlases.

Additionally, given a definable space X and a compatible definable atlas $(U_i, \phi_i)_{i \in I}$, we say that X is a *definable n-manifold* or a *definable manifold of dimension n* if for each $i \in I$, $\phi_i(U_i) \subseteq M^n$ is open.

In particular, note that any definable manifold is also a definable space.

Example 2.3.7.

- *Let $X \subseteq M^n$ be a definable set. Then X with the induced topology is a definable space, where a representative for the equivalence class of definable atlases is the trivial definable atlas $\{(X, id)\}$. Furthermore, if X is open in M^n, then it would be a definable n-manifold.*

- *One important example we have already seen is that of definable groups. If G is a group definable in $\mathcal{M}$, then the t-topology from Theorem 2.2.8 makes G a definable n-manifold.*

Additionally, note that the same set can admit both a definable space structure and admit a definable manifold structure that are not equivalent.

Example 2.3.8. *Let $\mathcal{M} = (\mathbb{R}, +, \cdot, 0, 1)$ and let $X = [0, 1)$. Let $\{(U_1, \phi_1), (U_2, \phi_2)\}$ be a definable atlas given by:*

- $U_1 = \{0\}$ *and* $\phi_1 : \{0\} \to \{0\}$ *is the identity;*
- $U_2 = (0, 1)$ *and* $\phi_2 : (0, 1) \to (0, 1)$ *is the identity;*

Then X equipped with the definable atlas $(U_i, \phi_i)_{i=1,2}$ is a definable space, but not a definable manifold.

On the other hand, consider the definable atlas $\{(V_1, \psi_1), (V_2, \psi_2)\}$ given by:

- $V_1 = [0, 0.1) \cup (0.9, 1)$ *and* $\psi_1 : V_1 \to (0, 0.2)$ *given by*

$$\psi_1(t) = \begin{cases} t + 0.1 & \textit{if } t \in [0, 0.1) \\ t - 0.9 & \textit{if } t \in (0.9, 1) \end{cases}$$

- $V_2 = (0, 1)$ *and* $\psi_2 : (0, 1) \to (0, 1)$ *is the identity;*

Then X equipped with the definable atlas $(V_i, \psi_i)_{i=1,2}$ is a definable 1-manifold.

In particular, note that $(V_i, \psi_i)_{i=1,2}$ and $(U_i, \phi_i)_{i=1,2}$ induce different topologies on X: X equipped with $(U_i, \phi_i)_{i=1,2}$ has 0 as an isolated point, while X equipped with $(U_i, \phi_i)_{i=1,2}$ is homeomorphic to $\mathbb{S}^1$ and thus has no isolated points.

Recall that a topological space X is said to be regular if for any $a \in X$ and closed set $F \subseteq X$ with $a \notin F$, there are disjoint open sets U and V such that $a \in U$ and $F \subseteq V$.

In particular, a definable space X is regular if and only if for any $a \in X$ and $F \subseteq X$ definable and closed, there are disjoint definable open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 3.5 of [16], expands on this in the following way.

Lemma 2.3.9. *Any Hausdorff definable manifold is regular.*

The last thing I want to discuss in this chapter is definable compactness. The notion of definable compactness was introduced in [10] with the aim of providing a notion akin to that of classical topological compactness, but more suitable in the context of o-minimality.

Definition 2.3.10. Let X be a definable space. We say that X is *definably compact* if for every definable curve $\sigma : (a, b) \to X$, with $(a, b) \subseteq M$, there exist $\alpha, \beta \in X$ such that $\lim_{x \to a^+} \sigma(x) = \alpha$ and $\lim_{x \to b^-} \sigma(x) = \beta$.

Note that to show that a definable space X is definably compact, it is enough to show that for any definable curve $\sigma : (a,b) \to X$, at least one of the limits $\lim_{x\to a^+}$ and $\lim_{x\to b^-}$ exists.

Example 2.3.11. *Let $\mathcal{R} = (\mathbb{R}, +, \cdot, 0, 1)$ be the real field, let $\mathcal{R}^*$ be a sufficiently saturated elementary expansion of $\mathcal{R}$ and consider the interval $[0,1]$ in $\mathcal{R}^*$. This interval is definably compact as it is closed and bounded (we will see this shortly), but not compact. To see this, note that given any positive infinitesimal ϵ, then the open cover given by*

$$[0,\epsilon) \cup \{(x-\epsilon, x+\epsilon) : x \in (0,1)\} \cup (1-\epsilon, 1]$$

has no finite subcover.

In [10], Y. Peterzil and C. Steinhorn proved the following result.

Theorem 2.3.12. *Let $S \subseteq M^n$ be a definable subset with the subspace topology. Then S is definably compact if and only if S is bounded and closed.*

In particular, in any o-minimal expansion of $(\mathbb{R}, <)$, a definable set $S \subseteq \mathbb{R}^n$ is definably compact if and only if it is compact.

Y. Peterzil and A. Pillay provide another equivalent characterization of definable compactness in [11], using the familiar condition that any open cover has a finite subcover, with some necessary modifications to capture the definable aspect of the definition. This, along with other properties of definably compact spaces, will be explored in the next chapter.

3. Definable Semigroups in O-minimal Structures

3.1. General semigroup Theory

I will begin this chapter by reviewing some basic results from semigroup theory that we will use thoroughly in this chapter.

Recall that a *semigroup* is a pair $(S, \cdot)$ where S is non-empty and $\cdot$ is a binary associative operation on S. An element $e \in S$ is called an *idempotent* if $e^2 = e$, and the set of idempotents of S is denoted by *E(S)* . Given a semigroup S, we say that $I \subseteq S$ is a *right ideal* of S if it is non-empty and, for all $i \in I$ and $s \in S$, we have $is \in I$. *Left ideals* are defined analogously and a subset is an *ideal* if it is a right and left ideal.

A *monoid* M is a semigroup with an identity, i.e. there exists an element $e \in M$ such that for all $m \in M$, $em = me = m$. Given a semigroup S, we define its *induced monoid*, denoted by S^1 as: If S is already a monoid, then $S^1 = S$. If S is not a monoid, then consider the set $S \cup \{1\}$, where 1 is a new element and define the product $*$ in this set as:

$$s * t = \begin{cases} s \cdot t & \text{if } s, t \in S \\ s & \text{if } t = 1 \\ t & \text{if } s = 1 \end{cases}$$

With this notation, the left ideal generated by an element $s \in S$ is the subset $S^1 s$, the right ideal is sS^1 and the ideal generated by s is $S^1 s S^1$.

Definition 3.1.1. Let S be a semigroup. For each $s, t \in S$ define:

$$\begin{aligned} s \leq_{\mathscr{L}} t \quad &\text{if} \quad S^1 s \subseteq S^1 t \\ s \leq_{\mathscr{R}} t \quad &\text{if} \quad sS^1 \subseteq tS^1 \\ s \leq_{\mathscr{J}} t \quad &\text{if} \quad S^1 s S^1 \subseteq S^1 t S^1 \end{aligned}$$

Additionally, we define the following equivalence relations on $(S, \cdot)$.

$$\mathscr{L} = \leq_{\mathscr{L}} \cap \geq_{\mathscr{L}}$$
$$\mathscr{R} = \leq_{\mathscr{R}} \cap \geq_{\mathscr{R}}$$
$$\mathscr{J} = \leq_{\mathscr{J}} \cap \geq_{\mathscr{J}}$$

Note that these are definable equivalence relations on S. Furthermore, we define the equivalence relation $\mathscr{H}$ as $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$.

Given two equivalence relations σ, ρ on a set X, we define their composition $\sigma\rho$ as the equivalence relation given by

$$(x, y) \in \sigma\rho \Leftrightarrow \exists z \in X : (x, z) \in \sigma \wedge (z, y) \in \rho$$

Proposition 3.1.2. *Let S be a semigroup. Then $\mathscr{R}\mathscr{L} = \mathscr{L}\mathscr{R}$.*

For the proof, see Proposition 2.1.3 in [17].

We define the relation $\mathscr{D}$ on a semigroup S as $\mathscr{D} = \mathscr{R}\mathscr{L} = \mathscr{L}\mathscr{R}$, i.e. given $s, t \in S$, we have $s \mathrel{\mathscr{D}} t$ if and only if there is $c \in S$ such that $s \mathrel{\mathscr{L}} c \mathrel{\mathscr{R}} t$.

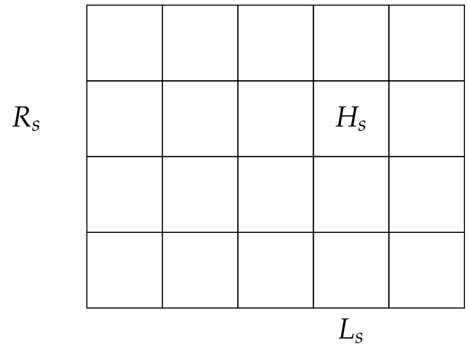
Another possible definition of $\mathscr{D}$ is given by the fact that $\mathscr{D}$ is the smallest equivalence relation containing both $\mathscr{R}$ and $\mathscr{L}$. In turn, this implies that $\mathscr{D} \subseteq \mathscr{J}$.

The equivalence relations $\mathscr{L}, \mathscr{R}, \mathscr{J}, \mathscr{H}$ and $\mathscr{D}$ are known as *Green's equivalence relations*, which can be depicted in the following Hasse diagram

$$\begin{tikzcd}
 & \mathscr{J} \arrow[d, no head] & \\
 & \mathscr{D} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathscr{L} \arrow[dr, no head] & & \mathscr{R} \arrow[dl, no head] \\
 & \mathscr{H} &
\end{tikzcd}$$

For a given $s \in S$, it is usual to denote by L_s, R_s, J_s, H_s and D_s the $\mathscr{L}, \mathscr{R}, \mathscr{J}, \mathscr{H}$ and $\mathscr{D}$ classes of s respectively.

Because inside a $\mathscr{D}$-class, any $\mathscr{R}$ and $\mathscr{L}$-classes have non-empty intersection, it is common to represent it as a diagram where the rows represent the $\mathscr{R}$-classes inside a given $\mathscr{D}$-class, and the columns represent an $\mathscr{L}$-class.



These kinds of diagrams are known as eggbox diagrams of a $\mathscr{D}$-class.

Note that $\mathscr{R}$ is a left congruence, i.e. given $s \in S$ and $a \mathrel{\mathscr{R}} b$, we have that $sa \mathrel{\mathscr{R}} sb$. Analogously, $\mathscr{L}$ is a right congruence.

The following two results are essential when working with Green's relations.

Lemma 3.1.3 (Green's Lemma)**.** *Let S be a semigroup.*

1. *Let $s, t \in S$ be such that $s \mathrel{\mathscr{L}} t$ and let $u, v \in S^1$ such that $us = t$ and $vt = s$. Then, the maps*

$$\phi : R_s \to R_t$$
$$x \mapsto ux$$

$$\psi : R_t \to R_s$$
$$x \mapsto vx$$

are mutually inverse bijections. Furthermore, for any $x, x' \in R_s$, $x \mathrel{\mathscr{H}} x'$ if and only if $\phi(x) \mathrel{\mathscr{H}} \phi(x')$.

2. *Let $s, t \in S$ such that $s \mathrel{\mathscr{R}} t$ and let $u, v \in S^1$ such that $su = t$ and $tv = s$. Then, the maps*

$$\phi : L_s \to L_t$$
$$x \mapsto xu$$

$$\psi : L_t \to L_s$$
$$x \mapsto xv$$

are mutually inverse bijections. Furthermore, for any $x, x' \in L_s$, $x \mathscr{H} x'$ if and only if $\phi(x) \mathscr{H} \phi(x')$.

3. *Let $s, t \in S$ such that $s \mathscr{D} t$, i.e. there exists $r \in S$ such that $s \mathscr{L} r \mathscr{R} t$ and let $u, v, w, z \in S^1$ such that $us = r$, $vr = s$, $rw = t$ and $tz = r$. Then, the maps*

$$\phi : H_s \to H_t$$
$$x \mapsto uxw$$

$$\psi : H_t \to H_s$$
$$x \mapsto vxz$$

are mutually inverse bijections.

Proposition 3.1.4. *Let S be a semigroup and $a, b \in S$ be two elements in the same $\mathscr{D}$-class. Then, $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.*

For the proof of these two results, see Sections 2.2 and 2.3 of [17] respectively.

Note that any subgroup of a semigroup S is contained in a single $\mathscr{H}$-class. This next theorem, due to Green, gives equivalent conditions for an $\mathscr{H}$-class to be itself a subgroup.

Theorem 3.1.5. *[Green's Theorem] Let S be a semigroup and $s \in S$. The following are equivalent:*

(1) H_s is a group;

(2) H_s contains an idempotent;

(3) $s^2 \in H_s$;

(4) there exist $x, y \in H_s$ such that $xy \in H_s$.

Note that by Green's Lemma, all group $\mathscr{H}$-classes inside the same $\mathscr{D}$-class are isomorphic. For this, see Proposition 2.3.6 in [17].

By a *simple semigroup*, we mean a semigroup with no proper ideals.

The following is a useful alternative characterization of simple semigroups.

Lemma 3.1.6. *Let S be a semigroup. Then S is simple if and only if, for all $a \in S$, we have $SaS = S$.*

Proof. For any $a \in S$, SaS is an ideal. So S being simple implies that $SaS = S$.

On the other hand, assume that for all $a \in S$ we have that $SaS = S$. Let I be an ideal of S and fix $a \in I$. Then $S = SaS \subseteq SIS \subseteq I$, and this $S = I$. ☐

Given a semigroup S and $e, f \in E(S)$, we write $e \leq f$ if $ef = fe = e$. We call $\leq$ the *natural partial order* on $E(S)$.

Definition 3.1.7. Let S be a semigroup and $e \in E(S)$. We say that e is *primitive* if for all $f \in E(S), f \leq e \Rightarrow f = e$.

Definition 3.1.8. We say that a semigroup S is *completely simple* if it is simple and has at least one primitive idempotent.

Example 3.1.9.

- *Any group is completely simple;*
- *A semigroup S is called a* band *if $E(S) = S$. A band S is called a* rectangular band *if for all $a, b \in S$, we have $aba = a$. Any rectangular band is completely simple.*
- *If G is a group and B is a rectangular band, $G \times B$ is completely simple. Such semigroups are called* rectangular groups*.*

An important class of completely simple semigroups are Rees matrix semigroups, which I will introduce now.

Definition 3.1.10 (Rees Matrix Semigroup)**.** Let S be a semigroup, I, Λ be non-empty sets, and let $P : \Lambda \times I \to S$ be a function.

The *Rees matrix semigroup* $\mathscr{M}(I, S, \Lambda, P)$ is defined as the set $I \times S \times \Lambda$ with operation given by

$$(i, s, \lambda) \cdot (j, t, \mu) = (i, sP(\lambda, j)t, \mu)$$

When dealing with Rees matrix semigroups, it is usual to denote $P(\lambda, i)$ by $p_{\lambda i}$.

The importance of Rees matrix semigroups comes from the fact that they are the only completely simple semigroups, up to isomorphism.

Theorem 3.1.11 (Rees's Theorem)**.** *Let S be a semigroup. S is completely simple if and only if there exists a group G, non-empty sets I, Λ and a map $P : \Lambda \times I \to G$ such that $S \simeq \mathscr{M}(I, G, \Lambda, P)$.*

Later we will need the construction used in the proof of Rees's Theorem, and therefore we shall prove it here.

Before proving the theorem we need the following auxiliary lemmas.

Lemma 3.1.12. *Let S be a completely simple semigroup. Then, for all $a \in S$, we have that $R_a = aS$.*

Proof. Let e be a primitive idempotent of S.

We begin by proving that the result is true for e, i.e. $R_e = eS$.

Let $y \in R_e$. Then, there exists $w \in S$ such that $y = ew$ and thus $y \in eS$.

Now, let $a = es \in eS$, for some $s \in S$. Clearly $a \leq_{\mathscr{R}} e$, so all that is left to prove is that $e \leq_{\mathscr{R}} a$, i.e. that there exists $k \in S$ such that $e = ak$.

Note that

$$ea = ees = es = a$$

By Lemma 3.1.6, we know that $SaS = S$, i.e., there exist $z, t \in S$ such that $zat = e$. Define $x := eze$ and $y = te$. Then we have the following identities including x, y, e:

$$e = xay$$

$$ex = xe = x$$

$$ye = y$$

With this in mind, set $f := ayx$. Then

$$f^2 = ayxayx = ayex = ayx = f$$

meaning that f is an idempotent. Note also that

$$ef = eayx = ayx = f$$

$$fe = ayxe = ayx = f$$

so $f \leq e$. As e is a primitive idempotent, we get that $f = e$, i.e. $ayx = e$, proving that $e \leq_{\mathscr{R}} a$.

With $R_e = eS$ established, let $a \in S$.

By definition, we have that $R_a \subseteq aS$.

On the other hand, let $b \in aS$. By Lemma 3.1.6, $SeS = S$, and so there exists $z, t \in S$ such that $zet = a$, and thus $b = zeu$, for some $u \in S$.

Note that $eu, et \in eS = R_e$, meaning that $eu \mathrel{\mathscr{R}} et$. As $\mathscr{R}$ is a left congruence, we get that $zeu \mathrel{\mathscr{R}} zet$ which is equivalent to $b \mathrel{\mathscr{R}} a$, i.e. $b \in R_a$. □

With an analogous proof we can show the following lemma.

Lemma 3.1.13. *Let S be a completely simple semigroup. Then, for all $a \in S$, we have that $L_a = Sa$.*

Lemma 3.1.14. *Let S be a completely simple semigroup. Then, for all $a, b \in S$, we have that $ab \in R_a \cap L_b$*

Proof. Let $a, b \in S$. Then $ab \in aS \cap Sb$. By lemmas 3.1.13 and 3.1.12, we have that $ab \in R_a \cap L_b$. □

This implies the following interesting corollary.

Corollary 3.1.15. *Let S be a completely simple semigroup. Every $\mathscr{H}$-class of S is a group.*

Proof. Let H be an $\mathscr{H}$-class and $a \in H$. By Lemma 3.1.14, $a^2 \in R_a \cap L_a = H$. By Theorem 3.1.5, H is a group. □

In particular, in a completely simple semigroup, every $\mathscr{H}$-class has a unique idempotent.

The last piece we need to prove Rees's Theorem is the following observation.

Lemma 3.1.16. *Let S be a completely simple semigroup. Then $\mathscr{D} = \mathscr{J}$. In particular, all elements of S are in the same $\mathscr{D}$-class.*

Proof. Let $a, b \in S$ and note that $ab \in aS \cap Sb = R_a \cap L_b$. Then $a \mathrel{\mathscr{R}} ab \mathrel{\mathscr{L}} b$, meaning that $a \mathrel{\mathscr{D}} b$. □

We can now easily prove Rees's Theorem.

Proof. Let S be a completely simple semigroup, and fix an idempotent $e \in S$. Let $G = H_e$, $I = L_e \cap E(S)$, $\Lambda = R_e \cap E(S)$ and $P : \Lambda \times I \to G$ be given by $(\lambda, i) \mapsto \lambda i$.

I claim that $S \simeq \mathscr{M}(I, G, \Lambda, P)$, with isomorphism given by

$$\begin{aligned} \phi : \mathscr{M}(I, G, \Lambda, P) &\to S \\ (i, g, \lambda) &\mapsto ig\lambda \end{aligned}$$

Start by noting that given $\lambda \in \Lambda$ and $i \in I$, then $\lambda i \in R_\lambda \cap L_i = R_e \cap L_e = H_e = G$, so P is indeed a function with co-domain G.

I will start by showing that ϕ is a semigroup homomorphism:

$$\begin{aligned} \phi((i, g, \lambda)(j, h, \mu)) &= \phi(i, gP_{\lambda,i}h, \mu) \\ &= igP_{\lambda,i}h\mu \\ &= ig\lambda ih\mu \\ &= \phi(i, g, \lambda)\phi(j, h, \mu) \end{aligned}$$

For the bijectivity, fix an $\mathscr{H}$-class H and let $a \in H$. Let $i \in R_a \cap L_e \cap E(S) \subseteq I$ and $\lambda \in L_a \cap R_e \cap E(S) \subseteq \Lambda$ (note that by Corollary 3.1.15, every $\mathscr{H}$-class has an idempotent, which guarantees the existence of such i and λ). By Green's Lemma 3.1.3, the map $x \mapsto ix\lambda$ is a bijection from G to H, which means that every element of H may be written uniquely as $ig\lambda$, for some $g \in G$. From this, it follows that ϕ is bijective. □

Example 3.1.17. *Following the examples given in Example 3.1.9, we have that:*

- *Given any group G, then $G \simeq \mathscr{M}(I, G, \Lambda, P)$, where $I = \Lambda = \{\bullet\}$ are sets with just one element, and $P : \Lambda \times I \to G$ maps $(\bullet, \bullet) \mapsto e$;*
- *Given a rectangular band B, it is a well known fact in semigroup theory that there are sets I and Λ such that $B \simeq I \times \Lambda$ where the product on $I \times \Lambda$ is given by*

  $$(i, \lambda)(j, \mu) = (i, \mu)$$

  In fact, some authors define rectangular bands as being semigroups of the form $I \times \Lambda$ with this multiplication. If we take $G = \{e\}$ to be the trivial group, then $B \simeq \mathscr{M}(I, G, \Lambda, P)$ where P is the only map from $\Lambda \times I$ to G;

- *Given a rectangular group $G \times B$, then $G \times B \simeq \mathscr{M}(I, G, \Lambda, P)$, where $B \simeq I \times \Lambda$ as described in the previous item, and P maps any element of $\Lambda \times I$ to e.*

3.2. Preliminary Results on Definably Compact Definable Spaces

We start by fixing the following convention:

> Unless stated otherwise, until the end of the chapter, $\mathcal{M}$ will be a sufficiently saturated o-minimal structure with definable Skolem functions.

Before continuing, recall that any structure with definable choice also has definable Skolem functions (see section 1.9). Surprisingly, in an o-minimal structure, the converse is also true. In [18], the authors proved an o-minimal structure $\mathcal{M}$ having definable Skolem functions is equivalent to $\mathcal{M}$ having definable choice.

The aim of this section is to prove some additional results about definable compactness that we will use extensively.

Lemma 3.2.1. *Let X be a definably compact Hausdorff definable space and $Y \subseteq X$ be a definable subset. Then Y is definably compact if and only if it is closed in X.*

Proof. By Corollary 2.9 of [19], if Y is definably compact, then it is closed.

On the other hand, assume that Y is closed and let $\gamma : (a, b) \to Y$ be a definable map. As X is definably compact, there exists $\beta \in X$ such that $\lim_{t \to b} \gamma(t) = \beta$. Then $\beta \in \overline{Y} = Y$. □

Lemma 3.2.2. *Let $f : X \to Y$ be a continuous definable map between definable spaces. If X is definably compact, then $f(X)$ is definably compact.*

Proof. Let $\gamma : (a, b) \to f(X)$ be a definable map. For each $s \in (a, b)$, let $H_s = f^{-1}(\gamma(\{s\}))$, and consider the definable family $\{H_s : s \in (a, b)\}$. As $\mathcal{M}$ has definable Skolem functions, there exists a definable map $\sigma : (a, b) \to X$ such that $\sigma(s) \in H_s$. In particular, given $s \in (a, b)$, $\sigma(s) \in H_s$ implies that $\sigma(s) \in f^{-1}(\gamma(\{s\}))$, i.e. $f(\sigma(s)) = \gamma(s)$. As X is definably compact, there exists $\beta \in X$ such that $\lim_{t \to b} \sigma(t) = \beta$. I claim that $\lim_{t \to b} \gamma(t) = f(\beta)$. Let U be an open neighborhood of $f(\beta)$. Then $f^{-1}(U)$ is an open neighborhood of β, meaning that there exists $s \in (a, b)$ such that $\sigma((s, b)) \subseteq f^{-1}(U)$. Then, $f(\sigma((s, b))) \subseteq f(f^{-1}(U)) \subseteq U$, and as $f(\sigma((s, b)))$ is simply $\gamma((s, b))$, we obtain the desired result. □

Lemma 3.2.3. *Let X and Y be two definable spaces. Then $X \times Y$ is definably compact if and only if both X and Y are definably compact.*

Proof. Assume that $X \times Y$ is definably compact and let $\pi_X : X \times Y \to X$ be the projection onto X. As $X = \pi_X(X \times Y)$, by Lemma 3.2.2 X is definably compact. A similar argument shows that Y is definably compact.

On the other hand, assume that both X and Y are definably compact, and let $\gamma : (a, b) \to X \times Y$ be a definable map. Composing with the projections onto X and Y, we get two definable maps $\gamma_X : (a, b) \to X$ and $\gamma_Y : (a, b) \to Y$. By definable compactness, there exists $\beta_X \in X$ and $\beta_Y \in Y$ such that $\lim_{t\to b} \gamma_X(t) = \beta_X$ and $\lim_{t\to b} \gamma_Y(t) = \beta_Y$. Using the fact that $\gamma = (\gamma_X, \gamma_Y)$, and that the projections onto X and Y are open, it is straightforward to see that $\lim_{t\to b} \gamma(t) = (\beta_X, \beta_Y)$. $\square$

We will now give a useful alternative characterization of definable compactness. On [11], Y. Peterzil and A. Pillay introduce the following: Let $X \subseteq M^n$ be a definable set equipped with the topology induced by M^n, and $A \subseteq M$. A family $\{U_s : s \in P\}$ of definable open subsets of X is said to be a definable open cover of X parameterized by a complete type over A if the family is uniformly definable, covers X and P is the set of realizations of some complete m-type over A. In other words, there exists a formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $p \in S_m^A(\mathcal{M})$ such that $P = p(\mathcal{M}) := \{s \in M^n : s \models p\}$ and, for all $s \in P$, $U_s = \phi(\mathcal{M}, s)$.

The following is Theorem 2.3 in [11].

Theorem 3.2.4. *Let $X \subseteq M^n$ be an $\mathcal{M}_0$-definable set in $\mathcal{M}$, where $\mathcal{M}_0$ is a small elementary substructure of $\mathcal{M}$. The following are equivalent:*

i) X is definably compact;

ii) If $\{U_s : s \in P\}$ is a definable open cover of X parameterized by a complete type over $\mathcal{M}_0$, then it contains a finite subcover of X.

The goal for now is to generalize this to our setting before moving on. This generalization is achieved by closely following the arguments given in [12].

Let X be a definable space. We say that X is definably normal if, given any two disjoint definable closed sets $F_1, F_2 \subseteq X$, there are disjoint definable open sets U_1, U_2 such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$.

Lemma 3.2.5 (Shrinking Lemma). *Let D be a definable set with a Hausdorff definable space structure, Ω a definably compact definable subset of D and $U_1, \dots, U_k$ definable open subsets of D such that $\Omega = \bigcup_{i=1}^{k}(\Omega \cap U_i)$. Then, there are definable open sets $V_i \subseteq U_i$, for each $i = 1, \dots, k$ such that $\overline{V}_i \subseteq U_i$ and $\Omega = \bigcup_{i=1}^{k}(\Omega \cap V_i)$.*

Proof. Start by noting that by Theorem 2.12 of [19], Ω is definably normal. We follow the proof of the affine version of the Shrinking Lemma given in Chapter 6, Lemma 3.6 of [7]. The proof is by induction. Assume that for some $l < k$, the sets $V_1, \dots, V_l$ are already defined such that, for each $i = 1, \dots, l$, V_i is a definable open subset of D with $V_i \subseteq U_i$, $\overline{V}_i \subseteq U_i$ and $V_1, \dots, V_l, U_{l+1}, \dots, U_k$ covers Ω.

Consider the following two disjoint closed subsets of Ω

$$A = \Omega \setminus U_{l+1}$$

$$B = \Omega \setminus \left(\bigcup_{m \leq l} V_m \cup \bigcup_{j=l+2}^{k} U_j \right)$$

As Ω is definably normal, there disjoint definable open sets $A' \supseteq A$ and $B' \supseteq B$. Set V_{l+1} to be B'. $\square$

We can slightly improve this lemma in the following way.

Lemma 3.2.6. *Let D be a definable set with a Hausdorff definable space structure and let $(U_i, \phi_i)_{i=1,\dots,l}$ denote a definable atlas compatible with D. Given a definably compact definable subset Ω of D, there are definable open subsets $V_i \subseteq U_i$ for $i = 1, \dots, l$ such that, $\overline{V}_i \subseteq U_i$, $\Omega = \bigcup_{i=1}^{l}(\Omega \cap V_i)$ and $\phi_i(\Omega \cap \overline{V}_i)$ is a closed bounded subset of M^{n_i}.*

Proof. By Lemma 3.2.5 there are definable open subsets $V_i \subseteq U_i$ for $i = 1, \dots, l$ such that, $\overline{V}_i \subseteq U_i$, $\Omega = \bigcup_{i=1}^{l}(\Omega \cap V_i)$. Note that, for each $i = 1, \dots, l$, $\Omega \cap \overline{V}_i$ is a closed subset of Ω, and thus, by Proposition 3.2.1, definably compact. Note that, for each $i = 1, \dots, l$ ϕ_i is a homeomorphism. This implies that $\phi_i(\Omega \cap \overline{V}_i)$ is a definably compact subset of M^{n_i}, and thus by Theorem 2.3.12, it is a closed and bounded definable subset of M^{n_i}. $\square$

Given a saturated structure $\mathcal{M}$, a definable set X and a small elementary substructure $\mathcal{M}_0$, the set of $\mathcal{M}_0$-conjugates of X is the set

$$\{\sigma(X) : \sigma : \mathcal{M} \to \mathcal{M} \text{ is an automorphism that fixes } \mathcal{M}_0 \text{ pointwise}\}$$

Let $\phi(x,y)$ be a formula and $a \in \mathcal{M}$ a parameter such that $X = \phi(\mathcal{M},a)$. Given an automorphism $\sigma : \mathcal{M} \to \mathcal{M}$, then $\sigma(X) = \phi(\mathcal{M},\sigma(a))$. In particular, by Corollary 1.8.9, this means that the set of $\mathcal{M}_0$-conjugates of X is actually the set

$$\{X(t) : t \models \text{tp}^{\mathcal{M}}(a/M_0)\}$$

where $X(t)$ denotes $\phi(\mathcal{M},t)$.

With this in mind, we say that the set of $\mathcal{M}_0$-conjugates of X is finitely consistent if for every finite number of elements $t_1,\ldots,t_n \models \text{tp}^{\mathcal{M}}(a/M_0)$ the intersection $X(t_1) \cap \ldots \cap X(t_n)$ is non-empty.

Proposition 3.2.7. *Let D be a definable set with a Hausdorff definable space structure, such that D is defined over a small elementary substructure $\mathcal{M}_0$ and Ω be a definably compact definable subset of D also defined over $\mathcal{M}_0$. Given a definably compact definable subset $X \subseteq \Omega$, the following are equivalent:*

i) The set of $\mathcal{M}_0$-conjugates of X is finitely consistent;

ii) X has some point in $\mathcal{M}_0$.

Proof. Let $X = \phi(\mathcal{M},a)$ for some paramater $a \in \mathcal{M}$.

Start by assuming that X has a point in $\mathcal{M}_0$, and let that point be $b \in X$. Let $t \models \text{tp}^{\mathcal{M}}(a/\mathcal{M}_0)$. I claim that $b \in X(t)$. In fact, by Corollary 1.8.9, there exists an automorphism $\sigma : \mathcal{M} \to \mathcal{M}$ that fixes $\mathcal{M}_0$ pointwise such that $\sigma(t) = a$. So

$$b \in X(t) \Leftrightarrow \mathcal{M} \models \phi[b,t] \Leftrightarrow \mathcal{M} \models \phi[\sigma(b),\sigma(t)] \Leftrightarrow \mathcal{M} \models \phi[b,a] \Leftrightarrow b \in X$$

Thus, for any finite collection of elements $t_1,\ldots,t_n \models \text{tp}^{\mathcal{M}}(a/M_0)$, we have that b is in the intersection $X(t_1) \cap \ldots \cap X(t_n)$.

On the other hand, assume that the set of $\mathcal{M}_0$-conjugates of X is finitely consistent. Let $(U_i,\phi_i)_{i=1,\ldots,l}$ be a definable atlas of D defined also over $\mathcal{M}_0$. By Lemma 3.2.6, we can find open definable sets $V_i \subseteq U_i$ such that, for each $i = 1,\ldots,l$, $\overline{V}_i \subseteq U_i$, $\Omega = \bigcup_{i=1}^{l}(\Omega \cap V_i) = \bigcup_{i=1}^{l}(\Omega \cap \overline{V}_i)$ and, $\phi_i(\Omega \cap \overline{V}_i)$ is a closed and bounded subset of M^{n_i}. Now, for each $i = 1,\ldots,l$, let X_i be the set $V_i \cap X = \overline{V}_i \cap X$. We end up with a collection of closed definable sets X_i such that $X = \bigcup_{i=1}^{l} X_i$. Now, note that $\phi_i(X_i)$ is definably compact, and thus, by Theorem 2.3.12, closed and bounded.

Using Theorem 2.1 of [11], we conclude that the set of $\mathcal{M}_0$-conjugates of $\phi_i(X_i)$ is finitely consistent if and only if $\phi_i(X_i)$ has some point in $\mathcal{M}_0$. It is important to note that, in this paper, the authors state Theorem 2.1 assuming that $\mathcal{M}$ is a saturated o-minimal expansion of a real closed field. However, they remark that it is enough to assume that $\mathcal{M}$ is a saturated o-minimal structure with definable choice, which by Theorem 1.1 of [18], is the case in our setting.

Claim: For each $i = 1, \ldots, l$, the set of $\mathcal{M}_0$-conjugates of X_i is finitely consistent if and only if X_i has a point in $\mathcal{M}_0$.

Start by noting that ϕ_i is a definable bijection defined over $\mathcal{M}_0$. Consider the sentence ψ given by

$$\exists x(x \in X_i) \leftrightarrow \exists x(x \in \phi_i(X_i))$$

As X_i is non-empty, we have that $\mathcal{M} \models \psi$, and as $\mathcal{M}_0 \preceq \mathcal{M}$ is an elementary substructure of $\mathcal{M}$, we are able to conclude that $\mathcal{M}_0 \models \psi$, i.e. X_i has a point in $\mathcal{M}_0$ if and only if $\phi_i(X_i)$ has a point in $\mathcal{M}_0$.

On the other hand, note that the set of $\mathcal{M}_0$-conjugates of X_i is finitely consistent if and only if the set of $\mathcal{M}_0$-conjugates of $\phi_i(X_i)$ is finitely consistent. To see this, start by noting that given $t \models \text{tp}^{\mathcal{M}}(a/M_0)$, then:

$$\Big(\phi_i(X_i)\Big)(t) = \phi_i\Big(X_i(t)\Big)$$

And, in particular, given $t_1, \ldots, t_k \models \text{tp}^{\mathcal{M}}(a/M_0)$, we have that:

$$\bigcap_{j=1}^{k} \Big(\phi_i(X_i)\Big)(t_j) = \bigcap_{j=1}^{k} \phi_i\Big(X_i(t_j)\Big) = \phi_i\left(\bigcap_{j=1}^{k} X_i(t_j)\right)$$

where the last equality holds because each ϕ_i is bijective. This implies that $\bigcap_{j=1}^{k}(\phi_i(X_i))(t_j)$ is non-empty if and only if $\bigcap_{j=1}^{k} X_i(t_j)$ is non-empty.

These two observations, together with the fact that the set of $\mathcal{M}_0$-conjugates of $\phi_i(X_i)$ is finitely consistent if and only if $\phi_i(X_i)$ has some point in $\mathcal{M}_0$, proves the claim.

Note that to prove that X has some point in $\mathcal{M}_0$, it is enough to prove that X_i has some point in $\mathcal{M}_0$ for some i and therefore, by the claim we just proved, it is enough to show that the set of $\mathcal{M}_0$-conjugates of some X_i is finitely consistent.

For this, let a be some parameter in $\mathcal{M}$ over which X is defined.

By our assumption that the set of $\mathcal{M}_0$-conjugates of X is finitely consistent, the set $\{X(t) :$

$t \models \mathrm{tp}^{\mathcal{M}}(a/M_0)\}$ is a partial type over M, and thus we can extend it to a complete type p over M. Let b be a realization of p in some $|M|^+$-saturated elementary extension $\mathcal{M}'$ of $\mathcal{M}$. In particular,

$$b \in \bigcap\{X(t)(\mathcal{M}') : t \models \mathrm{tp}^{\mathcal{M}}(a/M_0)\} \subseteq X(\mathcal{M}') \subseteq \Omega(\mathcal{M}')$$

As $\Omega = \bigcup_{i=1}^{l}(\overline{V}_i \cap \Omega)$, we conclude that $\Omega(\mathcal{M}') = \bigcup_{i=1}^{l}(\overline{V}_i(\mathcal{M}') \cap \Omega(\mathcal{M}'))$, meaning that $b \in \overline{V}_i(\mathcal{M}') \cap \Omega(\mathcal{M}')$, for some $i = 1, \ldots, l$.

Since, for any $t \models \mathrm{tp}^{\mathcal{M}}(a/M_0)$, $X_i(t) = \overline{V}_i \cap X(t) = \overline{V}_i \cap \Omega \cap X(t)$, in particular, we have that $X_i(t)(\mathcal{M}') = \overline{V}_i(\mathcal{M}') \cap \Omega(\mathcal{M}') \cap X(t)(\mathcal{M}')$, which implies that

$$b \in \bigcap\{X_i(t)(\mathcal{M}') : t \models \mathrm{tp}^{\mathcal{M}}(a/M_0)\}$$

Now, let $t_1, \ldots, t_n \models \mathrm{tp}^{\mathcal{M}}(a/M_0)$. Then $\mathcal{M}' \models \exists b(b \in X_i(t_1) \wedge \ldots \wedge b \in X_i(t_n))$. As $\mathcal{M}$ is an elementary substructure of $\mathcal{M}'$, $\mathcal{M} \models \exists b(b \in X_i(t_1) \wedge \ldots \wedge b \in X_i(t_n))$, meaning that the set of $\mathcal{M}_0$-conjugates of X_i is finitely consistent. $\square$

This theorem has the following important corollary which we will need in the next subsection. Before that, recall that a family of sets $\{X_i : i \in I\}$ is said to have the finite intersection property (FIP) if the intersection of any finite number of elements X_i is non empty.

Corollary 3.2.8. *Let D be a definable set with a Hausdorff definable space structure, such that D is defined over a small elementary substructure $\mathcal{M}_0$. Let $X \subseteq D$ be an $\mathcal{M}_0$-definable subset and let $\{F_x : x \in X\}$ be family of definable closed subsets of D parameterized by X with the FIP. If D is definably compact, then there are $m_1, \ldots, m_k \in \mathcal{M}_0$ such that every F_x contains at least one of the m_i's.*

Proof. For the sake of contradiction, assume that for any finite subset $\Gamma \subseteq M_0$, there exists $x \in X$ such that $\Gamma \cap F_x = \emptyset$. For each $a \in M_0$, consider the $\mathcal{L}_{M_0}$-formula $\phi_a(x)$ given by $x \in X \wedge a \notin F_x$

Let $q(x) = \{\phi_a(x) : a \in M_0\}$ and note that this is a partial type over M_0. By saturation, there exists $x_0 \in X$ that realizes q, i.e. $M_0 \cap F_{x_0} = \emptyset$. As D is definably compact and F_{x_0} is closed, by Lemma 3.2.1, F_{x_0} is definably compact. By Theorem 3.2.7, this implies that the set of $\mathcal{M}_0$-conjugates of F_{x_0} is not finitely consistent, i.e. there exists $t_1, \ldots, t_k \models$

$\mathrm{tp}^{\mathcal{M}}(x_0/M_0)$ such that $F_{t_1} \cap \ldots \cap F_{t_k} = \emptyset$. This contradicts the fact that the family $\{F_x : x \in X\}$ has the FIP as for each $i = 1, \ldots, k$, we have that $t_i \in X$. □

Before continuing I would like to make the following remark:

In the following subsections, the only use of this corollary is to conclude that every downward directed definable family of non-empty closed sets has non-empty intersection when we are working inside a definably compact Hausdorff definable space. In fact, this is the only topological result we will need to prove almost all of the main results we will see next.

In [20], P. Guerrero proved that, in an o-minimal structure with definable choice, a definable topological space is definably compact if and only if every downward directed definable family of non-empty closed sets has non-empty intersection.

As such, many of the results we will prove can be extended from the setting of Hausdorff definable spaces to that of Hausdorff definable topological spaces. We restrict ourselves to the former for clarity, since most natural examples admit such structures, and because this generalization is mostly straightforward and does not provide any additional insights about the behaviour of definable semigroups. For these reasons, the generalization will not be pursued here.

Finally, another important consequence of Theorem 3.2.7, is the following alternative characterization of definable compactness which I hinted at earlier.

Proposition 3.2.9. *Let D be a definable set with a Hausdorff definable space structure, such that D is defined over a small elementary substructure $\mathcal{M}_0$ and X be a $\mathcal{M}_0$-definable subset of D. The following are equivalent:*

i) X is definably compact;

ii) Any definable open cover of X parameterized by a complete type over $\mathcal{M}_0$, has a finite subcover of X.

Proof. $(ii) \Rightarrow (i)$ Assume that X is not definably compact, and let $\gamma : (a,b) \to X$ be a definable map such that $\lim_{t\to 1} \gamma(t)$ does not exist, and by o-minimality, assume without loss of generality that γ is injective and continuous. For each $x \in (a,b)$, let $F_x = \gamma([x,b))$, which is closed in X. Furthermore, given $a < t < s < b$, $F_s \subset F_t$.

Consider the partial type $p(v)$ over M_0 given by $\{m < v < b : m \in M_0\}$. Let $p \in S_1^{\mathcal{M}}(M_0)$ be a complete type that extends it, and by saturation, there exists $s \in M$ that realizes

p. In particular, note that $s \in (a,b)$ but s is greater then any element of M_0 in (a,b). Let P be the set of realizations of p. Then $\{X \setminus F_s : s \in P\}$ is a definable open cover of X parameterized by a complete type over $\mathcal{M}_0$ without any finite subcover.

$(i) \Rightarrow (ii)$ Let X be definably compact and let $\{U_s : s \in P\}$ be a definable open cover of X parameterized by a complete type over $\mathcal{M}_0$. Assume, for the sake of contradiction, that $\{U_s : s \in P\}$ has no finite subcover and let $F_s = X \setminus U_s$, which are closed and by Proposition 3.2.1 definably compact. Then, for any $s_1, \ldots, s_n \in P$, $F_{s_1} \cap \ldots \cap F_{s_n} \neq \emptyset$, since otherwise, $U_{s_1}, \ldots, U_{s_n}$ would be a finite subcover of X. Let a be an element of M such that $P = \text{tp}^{\mathcal{M}}(a/M_0)$, and note that $\{F_s : s \in P\} = \{F_s : s \models \text{tp}^{\mathcal{M}}(a/M_0)\}$ is the set of $\mathcal{M}_0$-conjugates of F_a, which is finitely consistent. By Proposition 3.2.7, there exists some $b \in M_0$ such that $b \in \bigcap_{s \in P} F_s \subseteq X$, meaning that $\bigcup_{s \in P} U_s \neq X$. □

3.3. Definable Semigroups in O-minimal Structures

The goal of this section is both trying to generalize known results about definable groups in o-minimal structures, and known results about topological semigroups to definable semigroups in o-minimal structures. Before proceeding, recall our assumption that we are working inside a sufficiently saturated o-minimal structure $\mathcal{M}$ with definable choice.

By a definable semigroup in S we mean a definable set $S \subseteq M^n$ for some $n \geq 1$ and a definable function $\cdot : S \times S \to S$ such that $(S, \cdot)$ is a semigroup.

We start by making the following observation about how the dimensions of $\mathscr{R}, \mathscr{L}$ and $\mathscr{H}$-classes relate to each other, inside of a $\mathscr{D}$-class.

Proposition 3.3.1. *Let S be a definable semigroup. Then, in any $\mathscr{D}$-class of S, every $\mathscr{H}$-class has the same dimension, every $\mathscr{R}$-class has the same dimension and every $\mathscr{L}$-class has the same dimension.*

Additionally, fix a $\mathscr{D}$-class D and let $\dim(\mathscr{H})$ denote the dimension of any $\mathscr{H}$-class in D, and define analogously $\dim(\mathscr{L})$ and $\dim(\mathscr{R})$ to be the dimensions of any $\mathscr{L}$-class and $\mathscr{R}$-class in D. Then

$$\dim(D) = \dim(\mathscr{R}) + \dim(\mathscr{L}) - \dim(\mathscr{H})$$

Proof. Fix a $\mathscr{D}$-class D and let $s, t \in D$. From Green's Lemma (Lemma 3.1.3), there exists a definable bijection between L_s and L_t, R_s and R_t and between H_s and H_t, and thus they have the same dimension.

Note that $\mathcal{M}$ has elimination of imaginaries, and let R be an arbitrary $\mathscr{R}$-class in D. There exists a definable set which we can identify as $R/\mathscr{H}$ and a definable surjective function $f : R \to R/\mathscr{H}$ such that $s \mathscr{H} t$ if and only if $f(s) = f(t)$. Analogously, there exists a definable set which we can identify with $D/\mathscr{L}$ and a definable surjective function $g : D \to D/\mathscr{L}$ such that $s \mathscr{L} t$ if and only if $g(s) = g(t)$.

Consider the definable function $h : R/\mathscr{H} \to D/\mathscr{L}$ given by $(x, y) \in h \Leftrightarrow \exists z(f(z) = x \wedge g(z) = y)$.

Claim: h is injective.

Let $h(x) = h(y)$. By definition, there exist $z_1, z_2 \in S$ such that $z_1 \in f^{-1}(x) \cap g^{-1}(h(x))$ and $z_2 \in f^{-1}(y) \cap g^{-1}(h(x))$. Note that $g^{-1}(h(x))$ is an $\mathscr{L}$-class in D and both $f^{-1}(x)$ and $f^{-1}(y)$ are $\mathscr{H}$-classes in R. As any $\mathcal{L}$-class contains non-trivially one $\mathscr{H}$-class in R, we get that $f^{-1}(x) = f^{-1}(y)$ and thus $x = y$.

Claim: h is surjective.

Let $y \in D/\mathscr{L}$. The set $g^{-1}(y)$ is an $\mathscr{L}$-class in D and as such it intersects non-trivially exactly one $\mathscr{H}$-class in R. Let x be an element of that $\mathscr{H}$-class. Then $h(f(x)) = y$

As h is a definable bijective map, by Lemma 2.1.28 we get that

$$\dim(R/\mathscr{H}) = \dim(D/\mathscr{L})$$

By Lemma 2.1.33, this is equivalent to:

$$\begin{aligned} &\dim(\mathscr{R}) - \dim(\mathscr{H}) = \dim(D) - \dim(\mathscr{L}) \\ \Leftrightarrow &\dim(D) = \dim(\mathscr{R}) + \dim(\mathscr{L}) - \dim(\mathscr{H}) \end{aligned} \qquad \square$$

Next, we are going to look into definable completely simple semigroups.

Before continuing I would like to make the following remark: In the context of o-minimality, it would be more natural to study definably simple semigroups, i.e. semigroups with no proper definable ideals. It is evident that any simple semigroup is definably simple. However, in this case the converse is also true.

Proposition 3.3.2. *Let S be a semigroup definable in a structure $\mathcal{M}$ (not necessarily o-minimal). Then S is simple if and only if S is definably simple.*

Proof. If S is simple, then by definition, S is also definably simple.

Assume now that S is definably simple. For any $a \in S$, the ideal SaS is definable, and thus $SaS = S$. By Lemma 3.1.6, S is simple. $\square$

Now, let S be a completely simple definable semigroup. From Rees's Theorem, we know that $S \simeq \mathscr{M}(I, G, \Lambda, P)$ for some group G, sets I, Λ and $P : \Lambda \times I \to G$. The following shows that Rees's Theorem adapts well to the setting of completely simple definable semigroups.

Proposition 3.3.3 (Definable Rees's Theorem)**.** *Let $\mathcal{M}$ be an o-minimal structure and let S be a completely simple definable semigroup. Then there exists a definable group G, definable sets I, Λ and a definable map $P : \Lambda \times I \to G$ such that S is definably isomorphic to $\mathscr{M}(I, G, \Lambda, P)$.*

Proof. From the proof of Rees's Theorem given in section 3.1, given any idempotent $e \in S$, there exists an isomorphism from S to $\mathscr{M}(I, G, \Lambda, P)$, where $G = H_e$, $I = Se \cap E(S)$, $\Lambda = eS \cap E(S)$ and $P : \Lambda \times I \to G$ is given by $P(\lambda, i) = \lambda i$, which are clearly definable. The isomorphism is given by

$$\begin{aligned} R : \mathscr{M}(I, G, \Lambda, P) &\to S \\ (i, g, \lambda) &\mapsto ig\lambda \end{aligned}$$

Which again, is clearly definable. $\square$

3.3.1 Definably compact semigroups

The t-topology was the key to many advances in the study of groups definable in o-minimal structures since its introduction in [9]. It is only natural that, when studying semigroups definable in o-minimal structures, a topology with properties akin to the t-topology is desirable. As such, we fix the following assumption:

> For the rest of this chapter, unless stated otherwise, all following mentions of definable groups and definable semigroups in $\mathcal{M}$ are assumed to be equipped with a Hausdorff definable space structure that makes the operations continuous.

In the theory of topological semigroups, compactness is a fundamental property that allows us to conclude key results about the algebraic structure of the semigroup.

The aim of this subsection is thus to examine the behavior of definable semigroups S that are definably compact, and to see what results from classical topological semigroup theory we can recover in this setting.

3.3.1.1 Existence of idempotents

A classic result in the theory of topological semigroups is the fact that any Hausdorff compact topological semigroup has at least one idempotent (see Theorem 1.8 in [21]). As we will see now, definable compactness is enough to assure the existence of idempotents in S.

However, before proving the result, I would like to fix the following notation: Given a set X, we will use Δ_X to denote the diagonal $\{(x,x) : x \in X\} \subseteq X \times X$. When X is implicit from context, we will simply write Δ instead of Δ_X.

Proposition 3.3.4. *Let S be a definably compact semigroup. Then S has at least one idempotent.*

Proof. Let $\mathcal{M}_0$ be a small elementary substructure such that S is M_0-definable.

Start by assuming that S is commutative.

Claim: For each $s \in S$, sS is closed in S.

This comes from the fact that $\{s\} \times S$ is definably compact by Lemma 3.2.3, and as multiplication is continuous, by Lemma 3.2.2, sS is definably compact. By Lemma 3.2.1, sS is closed.

Consider the definable family $\{sS : s \in S\}$ closed subsets parameterized by S. Note also that, as S is commutative, for any $s_1, \ldots, s_k \in S$, we have that $s_1S \cap \ldots \cap s_kS \neq \emptyset$ as $s_1 \ldots s_kS \subseteq s_iS$ for $i = 1, \ldots, k$, i.e. the family as the FIP.

Claim: The intersection $\bigcap_{s\in S} sS$ is non-empty.

By Corollary 3.2.8, there exists $m_1, \ldots, m_k \in \mathcal{M}_0$ such that for any $s \in S$, sS contains some m_i. Note that the family $\{sS : s \in S\}$ is downward directed, i.e. for any $a, b \in S$, there exists $c \in S$ such that $aS \supseteq cS$ and $bS \supseteq cS$ (indeed, we may take $c = ab$). I claim that there exists $i = 1, \ldots, k$ such that $m_i \in sS$, for all $s \in S$. To see this, assume that this is not the case. For each $i = 1, \ldots, k$, let $s_i \in S$ such that $m_i \notin s_iS$. As $\{sS : s \in S\}$ is downward directed, there exists $x \in S$ such that $xS \subseteq s_iS$ for each $i = 1, \ldots, k$. Note that there exists $j = 1, \dots, k$ such that $m_j \in xS$, and in particular, $m_j \in s_jS$, which is a contradiction. As such, there exists $i = 1, \dots, k$ such that $m_i \in \bigcap_{s \in S} sS$.

Let $T = \bigcap_{s \in S} sS$ and fix $t \in T$. As T is a semigroup, $tT \subseteq T$. On the other hand,

$$tT = t \bigcap_{s \in S} sS = \bigcap_{s \in S} tsS \supseteq T$$

meaning that $tT = T$. This means that there exists $e \in T$ such that $te = t$ and $s \in T$ such that $ts = e$. So $e^2 = ets = tes = ts = e$.

Now, assume that S is a definably compact semigroup with $Z(S) \neq \emptyset$, where $Z(S) = \{x \in S : \forall y \in S, xy = yx\}$ is its center.

Note that $Z(S)$ is a commutative definable subsemigroup of S. Indeed, if $x, y \in Z(S)$, then for any $s \in S$, we have $xys = xsy = sxy$, so $xy \in Z(S)$.

Furthermore, note that $Z(S)$ is closed in S. To see this, let $(s_i)_{i \in I}$ be a net on $Z(S)$ such that $\lim s_i = c$. Then, for any $t \in S$ we have

$$ct = (\lim s_i)t = \lim(s_it) = \lim(ts_i) = t(\lim s_i) = tc$$

and thus $c \in Z(S)$.

By Lemma 3.2.1, $Z(S)$ is a commutative definably compact definable semigroup, and thus it has an idempotent as we have seen previously.

Finally, let S be an arbitrary definably compact semigroup.

Fix $a \in S$, and consider its centralizer $C(a) = \{x \in S : xa = ax\}$. Note that this is a definable semigroup, as if $x, y \in C(a)$, then $xya = xay = axy$. Consider the definable continuous map $f_a : S \to S \times S$ given by $f(x) = (ax, xa)$. Then $C(a) = f_a^{-1}(\Delta)$. As S is Hausdorff, Δ is closed and thus $C(a)$ is a closed subsemigroup of S. By Lemma 3.2.1, $C(a)$ is definably compact. Furthermore, note that, by definition, $Z(C(a)) \neq \emptyset$ as $a \in Z(C(a))$, which implies that $C(a)$ has at least one idempotent. $\square$

Let $(S, \cdot)$ be a topological semigroup. If S is also a group, i.e. S has inverses with respect to the product $\cdot$, it might be the case that the map $x \mapsto x^{-1}$ is not continuous, even though the product is. However, this is not possible if the topology in S is compact: any compact topological semigroup S that is algebraically a group, is a topological group (see Theorem 1.13 in [21]). As we will see now, definable compact semigroups exhibit the same behaviour.

Lemma 3.3.5. *Let S be a definably compact semigroup. If S is algebraically a group, then it is a definably compact group, i.e. the inverse map is also continuous.*

Proof. Let $m : S \times S \to S$ denote the multiplication in S, $i : S \to S$ the inverse function and e its identity.

Start by noting that the graph of the inverse function $\Gamma(i) = \{(x, x^{-1}) : x \in S\}$ is given by:

$$\Gamma(i) = \{(x, y) \in G \times G : m(x, y) = e\} = m^{-1}(\{e\})$$

By continuity of m, $\Gamma(i)$ is closed.

Let $C \subseteq S$ be a definable closed subset. Let $\pi_1 : S \times S \to S$ denote the projection into the first n components. Note that

$$i^{-1}(C) = \pi_1((S \times C) \cap \Gamma(i))$$

and $(S \times C) \cap \Gamma(i)$ is closed in $S \times S$ and hence definably compact by Lemma 3.2.1. By Lemma 3.2.2, $\pi_1((S \times C) \cap \Gamma(i)) = i^{-1}(C)$ is definably compact and again, by Lemma 3.2.1, closed.

As any closed set in S is the intersection of definable closed subsets of S, it follows that the pre-image of any closed set under i is closed, and hence, i is continuous. □

Proposition 3.3.6. *Let S be a definably compact definable monoid such that $E(S) = \{1\}$, where 1 is the identity of S. Then S is a definably compact group.*

Proof. Let $s \in S$. As we saw in the proof of 3.3.4, sS is a definably compact semigroup. By the same proposition, $sS \cap E(S) \neq \emptyset$, so that $1 \in sS$, thus s has a right inverse. Analogously, we can prove that it has a left inverse.

Moreover, the right and left inverses of s have to be equal. To see this, let l be the left inverse of s and r the right inverse. Then

$$l = l1 = lsr = 1r = r$$

Meaning that algebraically, S is a group.

The inverse operation is obviously definable, which implies that S is a definable group. By Lemma 3.3.5, S is a topological group with respect to its topology, and thus a definably compact definable group. $\square$

Note that in this proposition, definable compactness is necessary to conclude that S is a group. For example, let $\mathcal{M} = (\mathbb{R}, +, \cdot, 0, 1)$ and consider the definable semigroup $S = \{x \in \mathbb{R} : x \geq 1\}$ with the operation being given by the product. Then, S is a monoid whose only idempotent is 1, however, it is not a group.

Corollary 3.3.7. *Let G be a definable group. Then every definably compact subsemigroup of G is a subgroup.*

Proof. Let $S \subseteq G$ be a definably compact definable subsemigroup of G. By Proposition 3.3.4, S has an idempotent, and as the only idempotent in G is the identity, we have that $E(S) = \{e\}$. As $e \in S$, S is actually a monoid. By Proposition 3.3.6, S is a subgroup. $\square$

With this corollary, we can prove the following result, which is direct generalization of Theorem 5.1 in [11], where the authors assume that G is equipped with Pillay's t-topology while we assume a more general Hausdorff definable space structure on G.

Corollary 3.3.8. *Let G be a definable group, S be a definable subsemigroup of G, and let $\bar{S}$ denote the closure of S. If $\bar{S}$ is definably compact, then S is a subgroup of G.*

Additionally, if G itself is definably compact, then every definable subsemigroup of G is a subgroup.

Proof. Let G be a definable group, S be a definable subsemigroup such that $\bar{S}$ is definably compact.

Claim: $\bar{S}$ is a subsemigroup of G.

Let $m : G \times G \to G$ denote multiplication on G. Then

$$m(\bar{S} \times \bar{S}) = m(\overline{S \times S}) \subseteq \overline{m(S \times S)} \subseteq \bar{S}$$

So we conclude that $\bar{S} \cdot \bar{S} \subseteq \bar{S}$, i.e. $\bar{S}$ is a subsemigroup of G.

By Corollary 3.3.7, we get that $\bar{S}$ is a subgroup. By Theorem 1.8 of Chapter 4 in [7], we know that $\dim(\bar{S} \setminus S) < \dim S$, i.e. S is large in $\bar{S}$. Let $x \in \bar{S}$, and note that by Lemma 2.2.2, there exist generic elements a and b of $\bar{S}$ such that $x = a \cdot b$. If either a or b where in $\bar{S} \setminus S$, then we would have that $\dim(\bar{S} \setminus S) = \dim S$, which is a contradiction. As such, $a, b \in S$, meaning that:

$$\bar{S} \subseteq S \cdot S \subseteq S \subseteq \bar{S}$$

Therefore, $S = \bar{S}$ and as we saw, $\bar{S}$ is a subgroup.

Assume now that G is itself definably compact. As $\bar{S}$ is closed, it is definably compact and we can apply the first half of this corollary to conclude that S is a subgroup. □

Recall that a semigroup S is said to be right cancellative if for all $a, b, c \in S$, $ac = bc$ implies that $a = b$. Left cancellative semigroups are defined analogously and a cancellative semigroup is simply a right and left cancellative semigroup.

Corollary 3.3.9. *Let S be a cancellative definably compact semigroup. Then S is a definably compact group.*

Proof. By Proposition 3.3.4, S has some idempotent e, which we fix. For any $x \in S$, we have that $ex = e^2x = e(ex)$ and so $x = ex$. And analogously, $xe = xe^2 = (xe)e$ which implies that $x = xe$.

Therefore, e is an identity of S and $E(S) = \{e\}$, making S a definably compact monoid with only one idempotent. By Proposition 3.3.6, S is a definable group. Finally, by Lemma 3.3.5, S is a topological group with respect to its topology, and thus a definably compact definable group. □

As all groups are cancellative, we showed that a definably compact semigroup is a group if and only if it is cancellative.

Note again that both in Corollary 3.3.7 and Corollary 3.3.9, definable compactness is necessary. For example, take $\mathcal{M} = (\mathbb{R}, +, \cdot, 0, 1)$, let $G = (\mathbb{R}, +)$ and $S = (\mathbb{R}_{>0}, +)$. Then S is a definable subsemigroup of G but it is not a group. Additionally, S is cancellative but it is not a group.

3.3.1.2 Minimal ideals

Another key property of compact semigroups is that such semigroups always have minimal left and right ideals, and a unique minimal ideal (see Theorem 1.29 in [21]). As we will see now, this is also the case for definably compact semigroups.

We start by proving the existence of a unique minimal ideal.

Proposition 3.3.10. *Let S be a definably compact semigroup. Then S has a unique minimal ideal.*

Proof. Let S be M_0-definable, where $\mathcal{M}_0$ is a small elementary substructure of $\mathcal{M}$.

For each $a \in S$, let $J(a)$ denote the ideal SaS, which is closed by Lemmas 3.2.3, 3.2.2 and 3.2.1. Then $\{J(a) : a \in S\}$ is a definable family of closed subsets parameterized by S. Furthermore, given $a_1, \ldots, a_n \in S$, we have that $J(a_1 \ldots a_n) \subseteq J(a_1) \cap \ldots \cap J(a_n)$, which means that the family $\{J(a) : a \in S\}$ has the FIP and is downward directed. By Corollary 3.2.8, using an analogous argument to that in the proof of Proposition 3.3.4, we can conclude that the intersection $I = \bigcap\{J(a) : a \in S\}$ is non-empty, and thus an ideal.

Claim: I is a minimal ideal.

To see this, let $J \subseteq I$ be an ideal, and let $x \in J$. Then $SxS \subseteq J \subseteq I \subseteq SxS$, meaning that $J = I$.

For the uniqueness, let I, J be two minimal ideals. As $IJ \subseteq I \cap J$ we get that $I = IJ = J$. $\square$

Note that to guarantee the existence of a minimal ideal, definable compactness is necessary. For example, take $\mathcal{M} = (\mathbb{R}, +, \cdot, 0, 1)$ and consider the semigroup $S = (0, 1)$ with the usual product. For each $n \geq 2$, consider the ideal $I_n = (0, 1/n)$. These ideals constitute a strictly descending chain $I_2 \supset I_3 \supset \ldots$ with $\bigcap_{n \geq 2} I_n = \emptyset$, and as such, there is no minimal ideal K in S as that would imply that $K \subseteq I_n$ for all $n \geq 2$.

In the literature, the unique minimal ideal of a compact semigroup S is usually called the kernel of S, which we will denote by $K(S)$ or simply K when the semigroup S is implicit from context.

Before continuing, note that given $x \in K$, we have that $SxS = K$ by minimality, which implies that K is both closed and definable.

We now establish the existence of minimal left and right ideals, starting with some technical lemmas.

The following is a generalization of Wallace's Swelling lemma (see Theorem 1.9 in [21])

Lemma 3.3.11. *Let S be a definably compact semigroup, and A be a definable closed subset of S. Given $t \in A$, if $A \subseteq tA$, then $A = tA$.*

Proof. Fix $t \in A$ with $A \subseteq tA$, and let T denote the set $\{x \in S : tA \subseteq xA\}$. If $x, y \in T$, then $tA \subseteq xA \subseteq xtA \subseteq xyA$, which allows us to conclude that $xy \in T$, i.e. T is a definably subsemigroup of S.

Claim: T is closed.

By Lemma 3.2.1, it is enough to show that T is definably compact. Let $\gamma : (c, d) \to T$ be a definable curve, with $c, d \in M$. As S is definably compact, there exists $\beta \in S$ such that $\lim_{x \to d^-} = \beta$. I claim that $\beta \in T$, i.e. $tA \subseteq \beta A$. Fix $ta \in tA$ and start by noting that, for each $x \in (c, d)$, $tA \subseteq \gamma(x)A$, so that $ta \in \gamma(x)A$. For each $x \in (c, d)$, let $A_x := \{y \in A : ta = \gamma(x)y\}$, which is uniformly defined by the formula $\phi(y, x) = y \in A \wedge ta = \gamma(x)y$. As $\mathcal{M}$ has definable Skolem functions there exists a definable map $\sigma : (c, d) \to S$ such that $\sigma(x) \in A_x$ for all $x \in (c, d)$. Because S is definably compact, there exists $\alpha \in S$ such that $\lim_{x \to d^-} \sigma(x) = \alpha$. I claim that $ta = \beta\alpha$. For all $x \in (c, d)$, $ta = \gamma(x)\sigma(x)$, which implies that

$$ta = \lim_{x \to b^-} (\gamma(x)\sigma(x)) = \lim_{x \to b^-} \gamma(x) \lim_{x \to b^-} \sigma(x) = \beta\alpha$$

So $\beta \in T$ and therefore T is closed.

By Proposition 3.3.4, T has some idempotent e. By the definition of T, we have that $A \subseteq tA \subseteq eA$. Fix $a \in A$, then there exists $b \in A$ such that $a = eb$ and hence, $ea = e(eb) = eb = a$, which implies that $eA \subseteq A$. From the inclusion $A \subseteq tA \subseteq eA$ that we had previously, the equality $A = tA$ follows. $\square$

Proposition 3.3.12. *Let S be a definably compact semigroup and fix $e \in E(S)$. Then the following are equivalent:*

i) Se is a minimal left ideal;

ii) eSe is a group;

iii) eS is a minimal right ideal;

iv) SeS is the minimal ideal of S;

Proof. This is Theorem 1.23 of [21] with a few modifications.

Items $(i), (ii)$ and (iii) can be proven to be equivalent only using algebraic properties of S, and require no topology at all. The same goes for the proof that (iii) implies (iv). In these cases, see the proof of Theorem 1.23 of [21].

As for the proof that (iv) implies (iii), let R be a right ideal contained in eS. Fix $x \in R$ and note that $xS \subseteq R \subseteq eS$, meaning that $SxS \subseteq SeS$. By minimality, $SxS = SeS$, and therefore, $e = axb$ for some $a, b \in S$. Now note that

$$xS \subseteq R \subseteq eS = axbS \subseteq axS$$

As xS is a definable closed subset of S, by the Swelling Lemma (3.3.11), $xS = axS$, and by the previous chain of inclusions, it follows that $R = eS$, thus proving the minimality of eS. □

With this, we can finally conclude the following.

Corollary 3.3.13. *Let S be a definably compact semigroup. Then S has minimal right and left ideals. In fact, $E(K) \neq \emptyset$, and for each $e \in E(K)$ we have that:*

i) Se *is a minimal left ideal;*

ii) eSe *is a group;*

iii) eS *is a minimal right ideal;*

iv) $K = SeS$

Proof. K is a closed subsemigroup of S and thus, by Lemma 3.2.1, K is definably compact. By Proposition 3.3.4, K has some idempotent.

The rest follows easily from Proposition 3.3.12. □

Note that if R is a minimal right ideal, then given $x \in R$, we have $xS = R$ by minimally, meaning that R is both definable and closed. The same is true for any minimal left ideal.

For the sake of completeness, we will look into some semigroup theoretic properties of the kernel of a semigroup S. Most proofs only use the minimality of the kernel and algebraic properties of the semigroup, meaning that o-minimality and definable compactness change nothing and the ”standard” proof remains valid. In such cases, I will only leave a reference to the proof.

Proposition 3.3.14. *Let S be a definably compact semigroup. Then $K(S)$ is the union of all minimal left ideals, and the union of all minimal right ideals. Moreover, these unions are disjoint.*

Proof. See remark 2.4 of [22]. □

Proposition 3.3.15. *Let S be a definably compact semigroup. Then K is a disjoint union of definable subgroups of S.*

Proof. This is Corollary 2.7 of [22]. The only thing left to do is proving the definability of the subgroups. This comes from the fact the subgroups are the intersection of one minimal left ideal with a minimal right ideal, which are definable. □

Proposition 3.3.16. *Let S be a definably compact semigroup. Given a left ideal L, then L is a minimal left ideal if and only if $L = Se$, for some $e \in E(K)$. Analogously, a right ideal R is a minimal right ideal if and only if $R = eS$ for some $e \in E(K)$.*

Proof. See Corollary 2.8 of [22]. □

Proposition 3.3.17. *Let S be a definably compact semigroup. Then, the following are equivalent.*

i) K *is a group;*

ii) K *has only one idempotent;*

iii) S *has only one minimal left ideal and only one minimal right ideal.*

Proof. See Theorem 2.9 of [22]. □

In particular, as K is the union of the minimal left ideals or the union of the minimal right ideals, if there is only one of each, then they both equal K.

Proposition 3.3.18. *Let S be a definably compact semigroup and $e \in E(S)$. Then the following are equivalent;*

i) $e \in K$;

ii) eSe *is a group;*

iii) $K = SeS$;

iv) e *is a minimal idempotent in* S *with respect to the natural partial order.*

Proof. See Theorem 2.11 of [22]. □

Recall that a congruence σ on a semigroup S is an equivalence relation compatible with the group operation, i.e. if $(s, s') \in \sigma$ and $(t, t') \in \sigma$, then $(st, s't') \in \sigma$.

Given a semigroup S and a congruence σ, consider the set S/σ of equivalence classes with a binary operation given by $[s] \cdot [t] = [st]$. Then S/σ is a semigroup, known as the quotient of S over σ. If $I \subseteq S$ is an ideal, then $(I \times I) \cup \Delta$ is a congruence on S whose quotient we denote by S/I. This construction is known as the Rees quotient of S over I. Intuitively, the semigroup S/I is obtained from S by identifying all the elements of I.

If S is a topological semigroup and I is a closed ideal, then S/I is a topological semigroup with the quotient topology. There is a very useful way of characterizing the open sets in S/I: If $\pi : S \to S/I$ denotes the quotient map that sends each element of S to each equivalence class, then the open sets in S/I are precisely:

$$\{\pi(U) : U \subseteq S \text{ is open and either } U \cap I = \emptyset \text{ or } I \subseteq U\}$$

Proposition 3.3.19. *Let S be a definably compact definable semigroup and I a definable closed ideal. Then S/I definable. Furthermore, topologically S/I with the quotient topology is a definably compact Hausdorff definable space.*

Proof. **Step 1**: S/I is definable.

Fix an element $x \in I$ which I will denote by 0. Consider the definable set $S \setminus I \cup \{0\}$. Consider the formula $\phi(a, b, c)$ given by

$$\begin{aligned}
&(a = 0 \wedge c = 0)\vee \\
&(b = 0 \wedge c = 0)\vee \\
&(a \in S \setminus I \wedge b \in S \setminus I \wedge ((ab \in S \setminus I \wedge c = ab) \vee (ab \in I \wedge c = 0))
\end{aligned}$$

Then $\phi(a, b, c)$ defines a product $*$ in the following way: If either a or b is 0, then $ab = 0$. If both $a, b \in S \setminus I$, then $a * b = ab$ if $ab \in S \setminus I$ or $a * b = 0$ if $a * b \in I$, which is the operation in the Rees quotient S/I.

Step 2: S/I is a definable space.

Let $(U_i, \phi_i)_{i=1,\ldots,n}$ be a definable atlas of S. For each $i = 1, \ldots, n$ let $V_i = U_i \cap I^c$ and consider the family $(V_i, \phi_i|_{V_i})_{i=1,\ldots,n}$. Without loss of generality assume that $V_i \neq \emptyset$ for all $i = 1, \ldots, n$. Denoting $\pi(I)$ by 0, we get that the family $(\pi(V_i), \phi_i|_{V_i} \circ \pi^{-1})_{i=1,\ldots,n}$ is a definable atlas in $(S/I) \setminus \{0\}$. Adding an additional chart $\psi : \{0\} \to \{a\}$ for some $a \in M$, we get that S/I is a definable space.

Step 3: S/I is definably compact.

I will start by proving that S/I is Hausdorff. Let $a, b \in S/I$ with $a \neq b$. As each non-zero element in S/I is an equivalence class with only one element, I will treat a and b as both an element of S and as their equivalence classes in S/I. If neither a nor b is zero, as S is Hausdorff, there are open disjoint open sets U', V' such that $a \in U'$ and $b \in V'$. Let $U = U' \cap I^c$ and $V = V' \cap I^c$ and note that these are disjoint open sets that separate a and b in S. As $I \cap U$ and $I \cap V$ are empty, we have that $\pi(U)$ and $\pi(V)$ are disjoint open sets in S/I with $a \in \pi(U)$ and $b \in \pi(U)$.

On the other hand, let $a = 0$ and $b \neq 0$, i.e. $b \notin I$. By Lemma 2.3.9, there are open disjoint sets U and V such that $I \subseteq U$ and $b \in V$, therefore $\pi(U)$ and $\pi(V)$ are disjoint open sets in S/I that separate $a = 0$ and b.

As for the curve completion, let $\gamma : (a, b) \to S/I$ be a definable curve. Assume without loss of generality that $0 \notin \gamma((a, b))$. This induces a unique curve $\sigma : (a, b) \to S$ such that $\pi \circ \sigma = \gamma$. By definable compactness, there exists $\beta \in S$ such that $\lim_{t \to b^-} \sigma(t) = \beta$. Then,

$$\pi(\beta) = \pi(\lim_{t \to b^-} \sigma(t)) = \lim_{t \to b^-} \pi(\sigma(t)) = \lim_{t \to b^-} \gamma(t)$$

Which concludes the proof. □

Let S be a definably compact semigroup. Then K is a definably compact completely simple semigroup as we have seen. On the other hand, S/K is a definably compact semigroup with a zero. Therefore, the study of definably compact semigroups can be split up into the study of these two families of definable semigroups, which is why we now change our focus towards definably compact completely simple semigroups.

3.3.2 Definably compact completely simple semigroups

The following is straightforward to prove.

Lemma 3.3.20. *Let I, Λ be topological spaces, G a topological group and $P : \Lambda \times I \to G$ a continuous map. Then $\mathscr{M}(I, G, \Lambda, P)$ with the product topology on $I \times G \times \Lambda$ is a topological semigroup.*

Not all topologies on completely simple semigroups arise in this way. For instance, we could have a topology on $I \times G \times \Lambda$ that is not a product topology.

This prompts the following definition.

Definition 3.3.21. A completely simple semigroup S is said to be a *topological paragroup* if there are topological spaces I, Λ, a topological group G and a continuous map $P : \Lambda \times I \to G$ such that S is topologically isomorphic (i.e. there exists an isomorphism that is also a homeomorphism) to $\mathscr{M}(I, G, \Lambda, P)$ with the product topology on $I \times G \times \Lambda$.

Proposition 3.3.22. *Let S be a definably compact completely simple semigroup. Then S is isomorphic to $\mathscr{M}(I, G, \Lambda, P)$, where I, Λ are definable sets, each with a Hausdorff definable space structure, G a definable group with a Hausdorff definable space structure compatible with the group operations, P a definable continuous map, the topology on $\mathscr{M}(I, G, \Lambda, P)$ is the product topology and the topological isomorphism between S and $\mathscr{M}(I, G, \Lambda, P)$ is definable. In particular, S is a topological paragroup.*

Proof. Following the proof of Lemma 3.3.3, fix an idempotent $e \in S$ and let $I = Se \cap E(S)$, $\Lambda = eS \cap E(S)$, $G = H_e = eSe$ and $P : \Lambda \times I \to G$ given by $P(\lambda, i) = \lambda i$.

Note that restricting the topology on S to I, Λ and G yields a definable space structure on each of these sets.

Furthermore, note that as $G = eSe$, we have that G is definably compact. Additionally, with the topology induced by S, G is a topological semigroup. By Proposition 3.3.5, G is actually a topological group. Consider also I and Λ with the topologies induced by S, and consider $\mathscr{M}(I, G, \Lambda, P)$ with the product topology.

Recall that the isomorphism from $\mathscr{M}(I, G, \Lambda, P)$ to S is given by

$$\begin{aligned} \phi : \mathscr{M}(I, G, \Lambda, P) &\to S \\ (i, g, \lambda) &\to ig\lambda \end{aligned}$$

With its inverse being

$$\begin{aligned} \psi : S &\to \mathscr{M}(I, G, \Lambda, P) \\ s &\mapsto (s(ses)^{-1}, ses, (ese)^{-1}s) \end{aligned}$$

Where the inverses are being taken inside of $G = eSe$. By continuity of the product and of the inverse map, both ϕ and ψ are continuous. $\square$

Remarks:

- If S is a definably compact semigroup then it is kernel is closed. As a consequence it is a definably compact completely simple semigroup. By Proposition 3.3.22, the kernel is actually a topological paragroup;
- Additionally, note that by construction, S being definably compact implies that I, Λ and G are also definably compact.

Recall that in a semigroup S, an element s is said to be regular if there exists $t \in S$ such that $sts = s$. A semigroup is said to be regular if every element is regular. The following is Proposition 2.4.2 of [17].

Proposition 3.3.23. *Let S be a semigroup and T be a regular subsemigroup of S. Then*

$$\mathscr{L}^T = \mathscr{L}^S \cap (T \times T)$$
$$\mathscr{R}^T = \mathscr{R}^S \cap (T \times T)$$
$$\mathscr{H}^T = \mathscr{H}^S \cap (T \times T)$$

Where $\mathscr{K}^S$ and $\mathscr{K}^T$ denotes the green relation $\mathscr{K}$ in S and in T respectively, where $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{H}\}$.

With this, we can prove the following.

Proposition 3.3.24. *Let $S \simeq \mathscr{M}(I, G, \Lambda, P)$ be definably compact and T be a definable subsemigroup of S. Then T is completely simple.*

Proof. Let T be a definable subsemigroup of $S \simeq \mathscr{M}(I, G, \Lambda, P)$. Start by noting that for each $\mathscr{H}$-class H_e with $e \in E(S)$, $T \cap H_e$ is either a subsemigroup of H_e or the empty set, and in particular, if $T \cap H_e$ is not empty, then by Corollary 3.3.8, it is a subgroup of H_e. This implies that the elements of $E(T)$ are precisely the idempotents of S where $T \cap H_e$ is a group. As $T = \bigcup_{e \in E(S)} (T \cap H_e)$, we conclude that T is a union of groups and thus T is regular.

Claim: T is simple. Given $t \in T$, by Proposition 3.3.23, $H_t^T = H_t^S \cap T$, meaning that every $\mathscr{H}$-class of T is a group.

Let I be an ideal of T and note that if $I \cap H^T \neq \emptyset$ for some $\mathscr{H}$-class H^T of T, then $H^T \subseteq I$.

Let $x \in I$ and let $e_0 \in E(T)$ be an idempotent such that $x \in H_{e_0}^T$, and then $H_{e_0}^T \subseteq I$. Now, let $f \in E(T)$ be any other idempotent, and note that $fe_0f \in H_f^T$. As I is an ideal, $fe_0f \in I$

and as such, we conclude that $H_f^T \subseteq I$ for each $f \in E(T)$. As $T = \bigcup_{f\in E(T)} H_f^T$, we get that $I = T$ and thus T is simple.

Now, as S is completely simple, every idempotent of S is primitive, which in turn implies that every idempotent of T is primitive as well. □

Corollary 3.3.25. *Let $S \simeq \mathscr{M}(I,G,\Lambda,P)$ be definably compact and T be a definable subsemigroup of S. Then there are subsets $J \subseteq I$, $\Gamma \subseteq \Lambda$ and a definable subgroup $W \subseteq G$ such that $T \simeq \mathscr{M}(J,W,\Gamma,P|_{\Gamma\times J})$.*

Proof. Let $e \in E(T)$. Recall that by the proof of Lemma 3.3.3, we may take $I = Se \cap E(S)$, $\Lambda = eS \cap E(S)$, $G = H_e$ and $P : \Lambda \times I \to G$ to be the map given by $P(\lambda, i) = \lambda i$.

By Proposition 3.3.24, T is completely simple. As such, following the same construction, if we take $J = Te \cap E(T)$, $\Gamma = eT \cap E(T)$, $W = H_e^T$ (where H_e^T is the $\mathscr{H}$-class of e in T) and $P : \Gamma \times J \to W$ to be the map given by $Q(\gamma, j) = \gamma j$. Then, have that $T \simeq \mathscr{M}(J,W,\Gamma,Q)$. Obviously, $J \subseteq I$, $\Gamma \subseteq \Lambda$, $W \subseteq G$ and $Q = P|_{\Gamma\times J}$. □

With this corollary, we can generalize some of the results we saw about definable groups. However, as we will see shortly, most of them only work when both $|I|$ and $|\Lambda|$ are finite and the semigroup is definably compact.

Just some terminology before I state the result: we say that a definable semigroup has the descending chain condition DCC if any descending sequence of definable subsemigroups $T_1 \supseteq T_2 \supseteq \ldots$ eventually stabilizes.

Proposition 3.3.26. *Let $S \simeq \mathscr{M}(I,G,\Lambda,P)$ be a definably compact completely simple semigroup equipped with a definable manifold structure. The following are equivalent:*

(i) Both I and Λ are finite;

(ii) There are only finitely many definable subsemigroups T with $\dim T = \dim S$;

(iii) S has the DCC;

(iv) A subsemigroup T of S is open if and only if $\dim T = \dim S$;

(v) Any definable subsemigroup of S is closed.

Proof. $(i) \Rightarrow (ii)$ Assume that both I and Λ are finite. For any subsemigroup $T \simeq \mathscr{M}(J,W,\Gamma,P|_{\Gamma\times J})$, note that $\dim(T) = \dim(W)$. Fixing a definable subgroup $W \subseteq G$, there are at most $2^{|I|}2^{|\Lambda|}$ subsemigroups of S of the form $T \simeq \mathscr{M}(J, W, \Gamma, P|_{\Gamma\times J})$. This means that

$$|\{T : T \text{ is a definable subsemigroup of } S \text{ and } \dim T = \dim S\}| \leq$$
$$2^{|I|} \cdot |\{W : W \text{ is a definable subgroup of } G \text{ and } \dim W = \dim G\}| \cdot 2^{|\Lambda|}$$

By Theorem 2.2.17, we arrive at the desired conclusion.

$(ii) \Rightarrow (iii)$ This proof is identical to the one we did for Theorem 2.2.18.

$(iii) \Rightarrow (i)$ Assume that either I or Λ are infinite. Without loss of generality, I will assume that I is infinite. This means that we can find an infinite subset $\{i_1, i_2, \ldots\} \subseteq I$. For each $n \geq 1$, let $I_n = I \setminus \{i_1, \ldots, i_n\}$, which is definable. Then

$$\mathscr{M}(I_1, G, \Lambda, P|_{\Lambda\times I_1}) \supset \mathscr{M}(I_2, G, \Lambda, P|_{\Lambda\times I_2}) \supset \ldots$$

Is an infinite descending chain of definable subsemigroups that never stabilizes, thus S does not have the DCC.

$(i) \Rightarrow (iv)$ Let T be a definable subsemigroup and write $T = \mathscr{M}(J, W, \Gamma, P|_{\Gamma\times J})$ as per Corollary 3.3.25. Note that $\dim T = \dim(J \times W \times \Gamma) = \dim(J) + \dim(W) + \dim(\Gamma)$. As both J and Γ are finite, we get that $\dim T = \dim W$. On the other hand, $\dim S = \dim(I \times G \times \Lambda) = \dim G$ for the same reason.

Now, assume that T is open. As the projection maps are open, we get that W is open in G. By Lemma 2.2.16, we can conclude that $\dim W = \dim G$, and thus $\dim T = \dim S$. On the other hand, if $\dim T = \dim S$, then $\dim W = \dim G$. By Lemma 2.2.16, W is open in G. As both J and Γ are open, we get that $T = J \times W \times \Gamma$ is open.

$(iv) \Rightarrow (i)$ We will show that I is finite, as the proof that Λ is finite is analogous.

Claim: Let X be a definable subset of I. If $\dim X = \dim I$, then X is open.

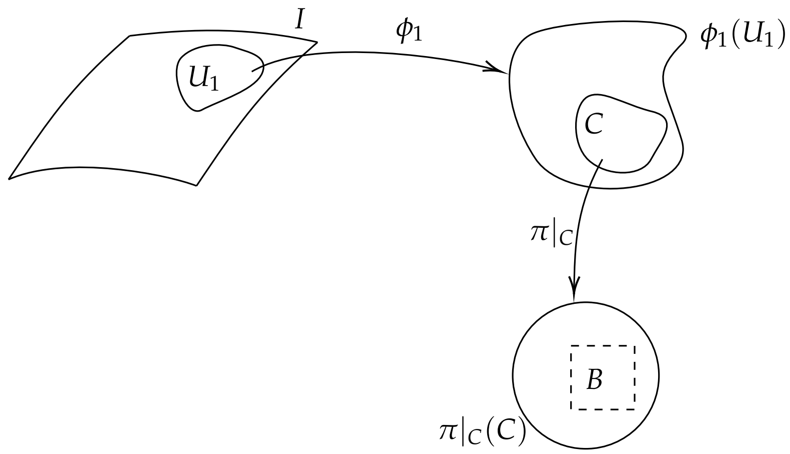
Let $X \subseteq I$ be a definable subset with $\dim X = \dim I$. Consider the definable semigroup $\mathscr{M}(X, G, \Lambda, P|_{\Lambda\times X})$. As $\dim(X \times G \times \Lambda) = \dim(S)$, by our hypothesis, $X \times G \times \Lambda$ is open, and as the projection map is open, X is open in I.

The claim is the property that will force the finiteness of I. For the sake of contradiction assume that I is infinite, i.e. $\dim(I) = d > 0$.

Let $\{(U_i, \phi_i)\}_{i=1,\ldots,k}$ be a definable atlas compatible with the definable space structure of

I. As $I = \bigcup_{i=1}^{k} U_i$, there exists some U_i with $\dim U_i = d$, which without loss of generality, we assume it to be U_1.

As ϕ_1 is a definable bijection, $\dim(\phi_1(U_1)) = d$. Let C be a cell of dimension d with $C \subseteq \phi_1(U_1)$ and let π be a projection onto M^d such that $\pi|_C : C \to \pi(C)$ is a homeomorphism into a cell of dimension d. Let $B = (a_1, b_1) \times \ldots \times (a_d, b_d)$ be some open box contained in $\pi|_C(C)$ (see image below).



Then, $\dim \pi|_C(C) \setminus B = \dim \pi|_C(C)$, and in particular, $\dim(I) = \dim(I \setminus (\pi|_C \circ \phi_1)^{-1}(B))$, but $I \setminus (\pi|_D \circ \phi_1)^{-1}(B)$ is not open in I, contradicting the claim we proved.

$(i) \Rightarrow (v)$ Let $T \subseteq S$ be a definable subsemigroup. By Corollary 3.3.25, there exists $J \subseteq I$, $\Gamma \subseteq \Lambda$ and a definable subgroup $W \subseteq G$ such that $T = \mathscr{M}(J, W, \Gamma, P|_{\Gamma \times J})$. By Proposition 2.2.13, W is closed, and as I and Λ have the discrete topology, J and Γ are closed. As a consequence, $T = J \times W \times \Gamma$ is closed.

$(v) \Rightarrow (i)$ Again, we will only show that I is finite as the proof for Λ is analogous. Let $x \in I$ and set $I_x = I \setminus \{x\}$. Let $T = \mathscr{M}(I_x, G, \Lambda, P|_{\Lambda \times I_x})$ be a definable subsemigroup of S. By our hypothesis, T is closed, and by Lemma 3.2.1, we conclude that T is definably compact. Projecting into I, we conclude by Lemma 3.2.2 that I_x is definably compact, and thus closed. This means that $\{x\}$ is open in I, and as $x \in I$ was arbitrary, we conclude that I is equipped with the discrete topology. Any definable space has only finitely many definably connected components, meaning that I has to be finite. $\square$

When a semigroup only has a finite number of $\mathscr{H}$-classes, we say that such semigroups are *$\mathscr{H}$-finite*. In particular, a completely simple semigroup being $\mathscr{H}$-finite is equivalent to both I and Λ being finite

Note that in Proposition 3.3.26, definable compactness is necessary. Consider for example the structure $\mathcal{M} = (\mathbb{R}, <, +, \cdot, 0, 1)$. Then $(\mathbb{R}, +)$ is a definable completely simple semigroup isomorphic to $\mathscr{M}(I, \mathbb{R}, \Lambda, P)$, where $I = \Lambda = \{0\}$ and $P(0,0) = 0$.

Then $(0, \infty) \supset (1, \infty) \supset \ldots \supset (n, \infty) \supset \ldots$ is a counterexample to both $(ii), (iii)$ and (v) in Proposition 3.3.26. Additionally, $[0, \infty)$ is a definable subsemigroup with the same dimension as $\mathbb{R}$ that is not open, thus being a counterexample to (iv) in Proposition 3.3.26.

Before proceeding, I would like to work though the following example.

Example 3.3.27. *We work inside the o-minimal structure $\mathcal{M} = (\mathbb{R}, +, \cdot, 0, 1)$. Consider the definable semigroup $S = [0,1]^2 \subseteq \mathbb{R}^2$ where the product is given by*

$$(a, b) * (c, d) = (ac, b)$$

which is definable. Let $S_1 = [0,1]$ with the operation given by $(r,s) \mapsto rs$ and $S_2 = [0,1]$ with operation given by $(r,s) \mapsto r$. Then $S = S_1 \times S_2$, so it is clear that S is a definably compact semigroup.

By Proposition 3.3.4, S has at least one idempotent, and indeed, it is easy to verify that

$$E(S) = \{0,1\} \times [0,1]$$

By Proposition 3.3.10, S has a minimal ideal, which is

$$K = \{0\} \times [0,1]$$

It is easy to see that K is an ideal, and for any ideal I, for any $(a,b) \in I$ and for any $(0,x) \in K$, we have that $(0,x)(a,b) = (0,x) \in I$, which implies that $K \subseteq I$.

Note that K has an infinite number of idempotents, which implies that K is not $\mathscr{H}$-finite. By Corollary 3.3.26, K does not have the DCC, which in turn implies that S does not have the DCC. Indeed

$$K = \{0\} \times [0,1] \supset \{0\} \times [0,1/2] \supset \{0\} \times [0,1/3] \supset \ldots$$

is an example of an infinite strictly descending chain of definable subsemigroups.

By Corollary 3.3.13, S also has minimal left and right ideals. Moreover, for any $e \in E(K) = K$, Se is a minimal left ideal and eS is a minimal right ideal, so we conclude that all minimal right ideals are of the form

$$\{(0,x)\} \text{ for } x \in [0,1]$$

and the only minimal left ideal is

$$\{0\} \times [0,1] = K$$

I would like end this chapter by making the following remarks:

Definable compactness seems, so far, to be a good generalization of the notion of compactness for studying definable semigroups, as foundational results in the theory of compact semigroups carry over to definably compact semigroups. However, there are some results I tried to prove in this more general setting with little success.

Namely, it is a known fact that, in any compact semigroup S, given any element $s \in S$, the topological closure of the subsemigroup generated by s has a unique idempotent. This prompts the following question:

Question. Let S be a definably compact semigroup. Given $s \in S$, is there a smallest definable subsemigroup $\langle s \rangle_{\text{def}}$ containing S? If so, is it true that $\overline{\langle s \rangle_{\text{def}}}$ has a unique idempotent?

Additionally, we say that a semigroup S is *right stable* if, for any $s, x \in S$,

$$s \mathscr{J} sx \Rightarrow s \mathscr{R} sx$$

and S is said to be *left stable* if, for any $s, x \in S$,

$$s \mathscr{J} xs \Rightarrow s \mathscr{L} xs$$

Furthermore, we say that S is *stable* if it is right and left stable.

It is known that any compact semigroup is stable, which raises the following natural question:

Question. Is it true that any definably compact definable semigroup is also stable?

At the time of writing this, I do not have an answer to these two questions.

Another interesting thing to note is that, contrary to the results about compact semigroups, results known to be true about definable groups do not, in general, carry over to this new context, as seen in Corollary 3.3.26.

A particular result about definable groups that I attempted to generalize without success is the fact that any group has a unique definable manifold structure that makes it a topological group (see Theorem 2.2.8).

This leads to the following question:

Question. Given an o-minimal structure $\mathcal{M}$, and a definable semigroup S, does there always exist a definable manifold structure on S that makes it a topological semigroup?

If the answer is positive, in general, said definable manifold structure will not be unique, unlike what happens with Pillay's t-topology on any definable group. To see this, let S be any definable set in $\mathcal{M}$, and fix any element of S which I will denote by 0. The operation $s * t = 0$ for any $s, t \in S$ is associative and definable, meaning that $(S, *)$ is a definable semigroup. Given any subset $X \subseteq S$, then the pre-image of X under $*$ is:

$$*^{-1}(X) = \begin{cases} \varnothing & \text{if } 0 \notin X \\ S \times S & \text{if } 0 \in X \end{cases}$$

which implies that any topology on S makes $*$ continuous, and in particular, any definable manifold structure in S will make S a topological semigroup.

References

[1] A. Tarski, “Contributions to the Theory of Models. I,” *Indagationes Mathematicae (Proceedings)*, vol. 57, pp. 572–581, Jan. 1954. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S1385725854500740 [Cited on page 1.]

[2] A. Pillay and C. Steinhorn, “Definable Sets in Ordered Structures. I,” *Transactions of the American Mathematical Society*, vol. 295, no. 2, pp. 565–592, 1986, publisher: American Mathematical Society. [Online]. Available: https://www.jstor.org/stable/2000052 [Cited on pages 1 and 2.]

[3] L. v. den Dries, “Remarks on Tarski’s problem concerning (R, +, *, exp),” *Studies in Logic and the Foundations of Mathematics*, vol. 112, no. C, pp. 97–121, Jan. 1984. [Online]. Available: http://www.scopus.com/inward/record.url?scp=77956967855&partnerID=8YFLogxK [Cited on page 1.]

[4] A. Grothendieck, *Esquisse d’un programme*. Chez l’auteur, 1984. [Cited on page 1.]

[5] D. Marker, *Model Theory: An Introduction*, ser. Graduate Texts in Mathematics. New York: Springer-Verlag, 2002, vol. 217. [Online]. Available: http://link.springer.com/10.1007/b98860 [Cited on pages 2, 23, 45, 47, 50, 59, 64, 66, and 71.]

[6] M. Edmundo, *Teoria de Modelos*. Faculdade de Ciências da Universidade de Lisboa, 2023. [Online]. Available: https://biblios.ciencias.ulisboa.pt/detalhes/65384 [Cited on page 2.]

[7] L. P. D. v. d. Dries, *Tame Topology and O-minimal Structures*, ser. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1998. [Online]. Available: https://www.cambridge.org/core/books/tame-topology-and-ominimal-structures/7AC940248AF2B05DA4D33E4FB05C97A2 [Cited on pages 2, 115, 130, and 141.]

[8] J. F. Knight, A. Pillay, and C. Steinhorn, “Definable Sets in Ordered Structures. II,” *Transactions of the American Mathematical Society*, vol. 295, no. 2, pp. 593–605, 1986, publisher: American Mathematical Society. [Online]. Available: https://www.jstor.org/stable/2000053 [Cited on pages 2, 80, and 84.]

[9] A. Pillay, "On groups and fields definable in *o*-minimal structures," *Journal of Pure and Applied Algebra*, vol. 53, no. 3, pp. 239–255, Sep. 1988. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0022404988901259 [Cited on pages 2, 104, 110, and 137.]

[10] C. Steinhorn and Y. Peterzil, "Definable Compactness and Definable Subgroups of o-Minimal Groups," *Journal of the London Mathematical Society*, vol. 59, Jul. 1996. [Cited on pages 2, 118, and 119.]

[11] Y. Peterzil and A. Pillay, "Generic sets in definably compact groups," *Fundamenta Mathematicae*, vol. 193, no. 2, pp. 153–170, 2007. [Online]. Available: https://eudml.org/doc/286617 [Cited on pages 2, 119, 129, 132, and 141.]

[12] M. J. Edmundo and G. Terzo, "A note on generic subsets of definable groups," *Fundamenta Mathematicae*, vol. 215, no. 1, pp. 53–65, 2011. [Online]. Available: https://eudml.org/doc/283329 [Cited on pages 3 and 129.]

[13] H.-D. Ebbinghaus and J. Flum, *Finite Model Theory*, ser. Springer Monographs in Mathematics. Berlin, Heidelberg: Springer, 1995. [Online]. Available: http://link.springer.com/10.1007/3-540-28788-4 [Cited on page 17.]

[14] A. J. Wilkie, "Model Completeness Results for Expansions of the Ordered Field of Real Numbers by Restricted Pfaffian Functions and the Exponential Function," *Journal of the American Mathematical Society*, vol. 9, no. 4, pp. 1051–1094, 1996, publisher: American Mathematical Society. [Online]. Available: https://www.jstor.org/stable/2152916 [Cited on page 71.]

[15] M. J. Edmundo, "Solvable groups definable in o-minimal structures," *Journal of Pure and Applied Algebra*, vol. 185, no. 1, pp. 103–145, Dec. 2003. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0022404903000859 [Cited on page 113.]

[16] M. Edmundo, "O-minimal cohomology and definably compact definable groups," Jan. 2001, arXiv:math/0012050. [Online]. Available: http://arxiv.org/abs/math/0012050 [Cited on page 118.]

[17] J. M. Howie, *Fundamentals of Semigroup Theory*, ser. London Mathematical Society Monographs. Oxford, New York: Oxford University Press, Feb. 1996. [Cited on pages 121, 123, and 150.]

[18] B. Dinis and M. J. Edmundo, “On definable Skolem functions and trichotomy,” *Annals of Pure and Applied Logic*, vol. 176, no. 10, p. 103632, Dec. 2025. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0168007225000818 [Cited on pages 128 and 132.]

[19] M. J. Edmundo, M. Mamino, and L. Prelli, “On definably proper maps,” *Fundamenta Mathematicae*, vol. 233, no. 1, pp. 1–36, 2016. [Online]. Available: https://eudml.org/doc/282893 [Cited on pages 128 and 130.]

[20] P. Andújar Guerrero, “Definable compactness in o-minimal structures,” *Model Theory*, vol. 4, no. 2, pp. 101–130, Mar. 2025, publisher: Mathematical Sciences Publishers. [Online]. Available: https://msp.org/mt/2025/4-2/p01.xhtml [Cited on page 134.]

[21] J. H. Carruth, J. A. Hildebrant, and R. J. Koch, *The Theory of Topological Semigroups*, ser. Monographs and Textbooks in Pure and Applied Mathematics. New York: Marcel Dekker, Inc., 1983, no. 75. [Cited on pages 138, 139, 142, 143, and 144.]

[22] K. Abodayeh and G. J. Murphy, “Compact Topological Semigroups,” *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, vol. 97A, no. 2, pp. 131–137, 1997, publisher: Royal Irish Academy. [Online]. Available: https://www.jstor.org/stable/20490240 [Cited on page 146.]

Index