Definable Semigroups in O-minimal Structures

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Setup

Fix a first-order language $\mathcal{L} = \{<, \dots\}$

Definition

An $\mathcal{L}$-structure $\mathcal{M} = (M, <, \dots)$ is said to be **o-minimal** if:

- $(M, <)$ is a dense linear order;
- Any definable subset of M is a finite union of points and intervals.

Assumption

We work inside a fixed sufficiently saturated o-minimal structure with definable choice

Definition

Let $\mathcal{M}$ be an o-minimal structure. A group $(G, \cdot)$ is said to be definable in $\mathcal{M}$ if $G \subseteq M^n$ is a definable set for some $n \geq 1$ and $\cdot : G \times G \to G$ is a definable map.
Definable semigroups are defined analogously.

Setup

Study definable semigroups in o-minimal structures

Known results about definable groups in o-minimal structures

Known results from semigroup theory

Setup

Definition

Let $X \subseteq M^n$ be a definable set. A **definable atlas** on X is a finite family $(U_i, \phi_i)_{i \in I}$ such that:

1. For each $i \in I$, $U_i \subseteq X$ is definable and $X = \bigcup_{i \in I} U_i$;
2. For each $i \in I$, $\phi_i : U_i \to \phi_i(U_i)$ is a definable bijection, where $\phi_i(U_i) \subseteq M^{n_i}$ is definable;
3. For each $i, j \in I$, each map $\phi_{ji} := \phi_j \circ \phi_i^{-1} := \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable homeomorphism.
4. For each $i, j \in I$, $\phi_i(U_i \cap U_j)$ is open in $\phi_i(U_i)$ with respect to the topology induced by $\mathcal{M}$;

Definition

Let $X \subseteq M^n$ be definable. A **definable** k**-manifold** structure on X is given by a definable atlas $(U_i, \phi_i)_{i \in I}$ where, for each $i \in I$, $\phi_i(U_i)$ is an open definable subset of M^k.

Setup

Theorem (A. Pillay)

Let G be a group definable in an o-minimal structure $\mathcal{M}$. There exists a unique definable manifold structure on G that makes G a topological group.

$$\downarrow$$

Assumption

We assume that every definable group/semigroup we work with is equipped with a Hausdorff definable space structure that makes it a topological group/semigroup.

Definition

Let X be a definable space. We say that X is **definably compact** if for every definable curve $\sigma : (a, b) \to X$, with $(a, b) \subseteq M$, there exists $\alpha, \beta \in X$ such that $\lim_{x \to a^+} \sigma(x) = \alpha$ and $\lim_{x \to b^-} \sigma(x) = \beta$.

Definably Compact Definable Spaces

Theorem (Y. Peterzil and A. Pillay)

Let $\mathcal{M}$ be a sufficiently saturated o-minimal structure with definable choice. Suppose X is a definable, closed and bounded subset of M^n, and that $\mathcal{M}_0$ is a small elementary substructure of M. Then the following are equivalent:

1. *The set of $\mathcal{M}_0$-conjugates of X is finitely consistent.*
2. *X has a point in $\mathcal{M}_0$*

Definably Compact Definable Spaces

Theorem (M. Edmundo and G. Terzo)

Let $\mathcal{M}$ be a sufficiently saturated o-minimal structure and let G be an $\mathcal{M}_0$-definable group, for some small elementary substructure $\mathcal{M}_0$. Consider Pillay's manifold topology on G and let Ω be a definably compact $\mathcal{M}_0$-definable subset of G, with respect to the topology induced by G. If X is a definably compact definable subset of Ω, then the following are equivalent:

1. *The set of $\mathcal{M}_0$-conjugates of X is finitely consistent.*
2. *X has a point in $\mathcal{M}_0$*

Proposition (3.2.7)

Let $\mathcal{M}$ be a sufficiently saturated o-minimal structure with definable choice, let D be a definable set with a Hausdorff definable space structure, such that D is defined over a small elementary substructure $\mathcal{M}_0$ and Ω be a definably compact definable subset of D also defined over $\mathcal{M}_0$. Given a definably compact subset $X \subseteq \Omega$, the following are equivalent:

i) *The set of $\mathcal{M}_0$-conjugates of X is finitely consistent;*

ii) *X has some point in $\mathcal{M}_0$.*

Existence of Idempotents

Proposition (3.3.4)

Let S be a definably compact semigroup. Then S has at least one idempotent.

Lemma (3.3.5)

Let S be a definably compact semigroup. If S is also algebraically a group, then S is a definably compact group, i.e. the map $s \mapsto s^{-1}$ is also continuous.

Existence of Idempotents: Consequences

Given a semigroup S, the set of idempotents of S is denoted by $E(S)$.
A **monoid** is a semigroup $(M, \cdot)$ that has an identity element, i.e. there exists $1 \in M$ such that for all $m \in M$, $1 \cdot m = m = m \cdot 1$.

Corollary (3.3.6)

Let M be a definably compact monoid with $E(M) = \{1\}$. Then M is a definably compact group.

Existence of Idempotents: Consequences

A semigroup S is said to be **left cancellative** if for any $a, b, c \in S$, $ab = ac$ implies that $b = c$. **Right cancellative** semigroups are defined analogously, and we say that a semigroup is **cancellative** if it is both right and left cancellative.

Note: Every group is cancellative, but the converse is not true:

Example

Fix $\mathcal{M} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$. Consider the definable semigroup $S = (\mathbb{R}_{>0}, +)$. Then S is cancellative, but it isn't a group.

Corollary (3.3.9)

Let S be a definably compact semigroup. Then S is a group if and only if S is cancellative.

Existence of Idempotents: Consequences

Theorem (Y. Peterzil and A. Pillay)

Let G be a definable group, S be a definable subsemigroup of G, and let $\bar{S}$ denote the closure of S. If $\bar{S}$ is definably compact, then S is a subgroup of G.
Additionally, if G itself is definably compact, then every definable subsemigroup of G is a subgroup.

$$\downarrow$$

Corollary (3.3.8)

Let G be a definable group, S be a definable subsemigroup of G, and let $\bar{S}$ denote the closure of S. If $\bar{S}$ is definably compact, then S is a subgroup of G.
Additionally, if G itself is definably compact, then every definable subsemigroup of G is a subgroup.

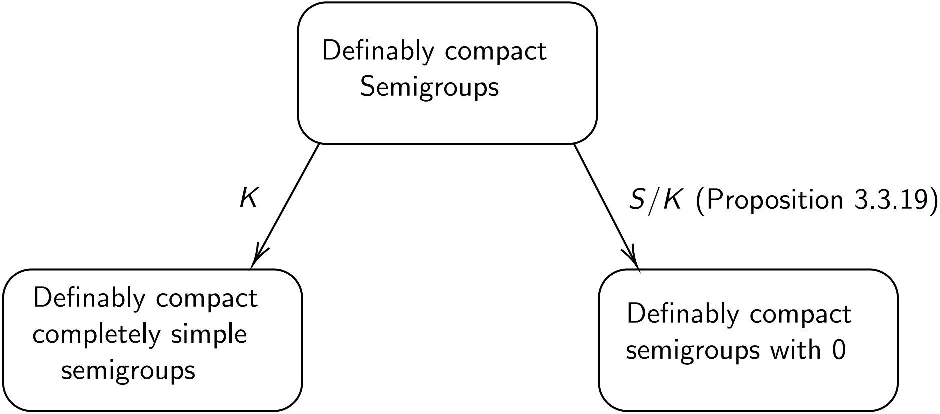
Minimal Ideals

Proposition (3.3.10 + 3.3.13)

Let S be a definably compact semigroup. Then S has minimal right ideals, minimal left ideals and a unique minimal ideal.
Furthermore all this minimal ideals are definable and definably compact.

Note:

- The unique minimal ideal of a compact semigroup S is denoted by $K(S)$ and it's called the **kernel** of S;
- $K(S)$ is completely simple;

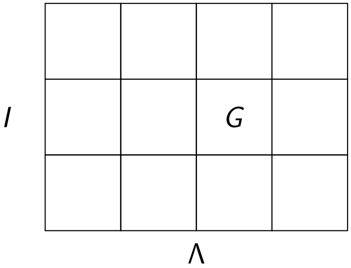

Definably compact Semigroups
K
S/K (Proposition 3.3.19)
Definably compact completely simple semigroups
Definably compact semigroups with 0

Definition (Rees Matrix Semigroup)

Let I, Λ be sets, G be a group and $P : \Lambda \times I \to G$ be a map. We define the semigroup $\mathscr{M}(I, G, \Lambda, P)$ as follows:

- Setwise, this semigroup is $I \times G \times \Lambda$;
- Given $(i, g, \lambda), (j, h, \mu)$ in $\mathscr{M}(I, G, \Lambda, P)$, we define:

$$(i, g, \lambda) \cdot (j, h, \mu) := (i, gP(\lambda, j)h, \mu)$$



Definable Completely Simple Semigroups

Theorem

For any sets I, Λ, group G and map $P : \Lambda \times I \to G$, the semigroup $\mathscr{M}(I, G, \Lambda, P)$ is completely simple.
Furthermore, if S is completely simple, then $S \simeq \mathscr{M}(I, G, \Lambda, P)$ for some Rees matrix semigroup.

$$\downarrow$$

Proposition (3.3.3)

Let S be a definable completely simple semigroup. There exist definable sets I, Λ, a definable group G, and a definable map $P : \Lambda \times I \to G$ such that S is definably isomorphic to $\mathscr{M}(I, G, \Lambda, P)$.

Definable Completely Simple Semigroups

Proposition

Let I, Λ be topological spaces, G a topological group and $P : \Lambda \times I \to G$ a continuous map. Then $\mathscr{M}(I, G, \Lambda, P)$ is a topological semigroup when we equip $I \times G \times \Lambda$ with the product topology.

Note: $\mathscr{M}(I, G, \Lambda, P)$ can be a topological semigroup, but with a topology that is not a product topology on $I \times G \times \Lambda$.

Definition

A topological completely simple semigroup S is said to be a topological paragroup if there are topological spaces I, Λ, a topological group G and a continuous map $P : \Lambda \times I \to G$ such that S is topologically isomorphic to $\mathscr{M}(I, G, \Lambda, P)$ with the product topology on $I \times G \times \Lambda$.

Definable Completely Simple Semigroups

Proposition (3.3.22)

Let S be a definably compact completely simple semigroup. Then S is a topological paragroup.
In particular, S is topologically isomorphic to $\mathscr{M}(I, G, \Lambda, P)$, where:

- *I and Λ are definable sets, each with a definably compact Hausdorff definable space structure;*
- *G is a definable group with a definably compact Hausdorff definable space structure compatible with the group operations*
- *$P : \Lambda \times I \to G$ a definable continuous map;*
- *the topology on $\mathscr{M}(I, G, \Lambda, P)$ is the product topology on $I \times G \times \Lambda$;*

Furthermore, the topological isomorphism between S and $\mathscr{M}(I, G, \Lambda, P)$ is definable.

Note: In particular, the kernel of a definably compact semigroup is a topological paragroup.

Definable Completely Simple Semigroups

Proposition (3.3.24 + 3.3.25)

Let $S \simeq \mathscr{M}(I, G, \Lambda, P)$ *be a definably compact completely simple semigroup and* T *be a definable sub-semigroup of* S*. Then* T *is completely simple.*
Furthermore, there are subsets $J \subseteq I$, $\Gamma \subseteq \Lambda$ *and a definable subgroup* $W \subseteq G$ *such that* $T \simeq \mathscr{M}(J, W, \Gamma, P|_{\Gamma \times J})$.

Definable Completely Simple Semigroups

Let G be a definable group. We say that G has the **descending chain condition** (DCC) if there is no infinite proper descending chain of definable subgroups of G.

Proposition

Let G be a definable group equipped with Pillay's manifold topology in some o-minimal structure. Then:

1. *There are only finitely many definable subgroups H of G with* $\dim H = \dim G$;
2. *G has the DCC.*
3. *A definable subgroup H of G is open if and only if* $\dim H = \dim G$;
4. *Any definable subgroup of G is closed;*

Definable Completely Simple Semigroups

Let S be a definable semigroup. We say that S has the **descending chain condition** if there is no infinite proper descending chain of definable sub-semigroups of G.

Proposition (3.3.26)

Let $S \simeq \mathscr{M}(I, G, \Lambda, P)$ be a definably compact completely simple semigroup equipped with a definable manifold structure. The following are equivalent:

1. *Both I and Λ are finite;*
2. *There are only finitely many definable sub-semigroups T of S with $\dim T = \dim S$;*
3. *S has the DCC.*
4. *A definable sub-semigroup T of S is open if and only if $\dim T = \dim S$;*
5. *Any definable sub-semigroup of S is closed;*

Definable Completely Simple Semigroups

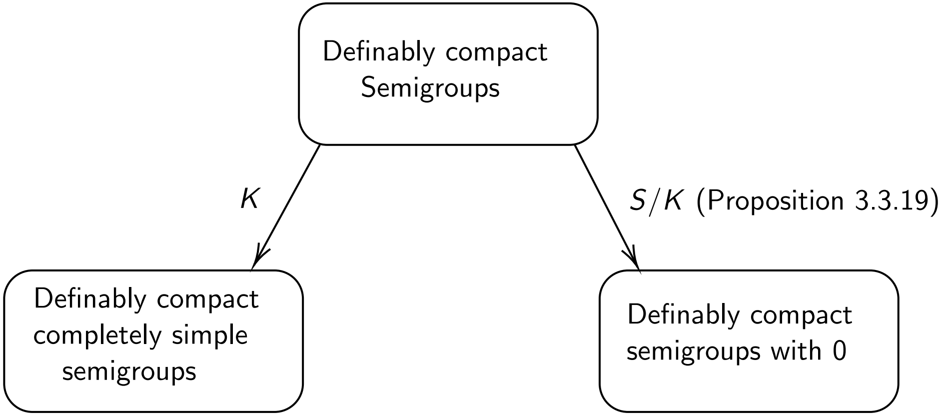
Note: In the last result, definable compactness is necessary.

Example

Fix $\mathcal{M} = (\mathbb{R}, <, +, \cdot, 0, 1)$.
Then $(\mathbb{R}, +)$ is a definable completely simple semigroup isomorphic to $\mathscr{M}(I, \mathbb{R}, \Lambda, P)$, where $I = \Lambda = \{0\}$ and $P(0,0) = 0$.

2. $(0, \infty) \supset (1, \infty) \supset \ldots \supset (n, \infty) \supset \ldots$
3. $(0, \infty) \supset (1, \infty) \supset \ldots \supset (n, \infty) \supset \ldots$
4. $[0, \infty)$
5. $(0, \infty)$

Where to go from here?



Where to go from here?

Question

Let S be a definably compact semigroup. Given $s \in S$, is there a smallest definable subsemigroup $\langle s \rangle_{\mathrm{def}}$ that contains s?
If so, is it true that $\overline{\langle s \rangle_{\mathrm{def}}}$ has a unique idempotent?

Where to go from here?

We say that a semigroup S is *right stable* if, for any $s, x \in S$,

$$s \mathrel{\mathscr{J}} sx \Rightarrow s \mathrel{\mathscr{R}} sx$$

and S is said to be *left stable* if, for any $s, x \in S$,

$$s \mathrel{\mathscr{J}} xs \Rightarrow s \mathrel{\mathscr{L}} xs$$

Furthermore, we say that S is *stable* if it is right and left stable.

Question

Is any definably compact semigroup S necessarily stable?

Where to go from here?

Question

Given an o-minimal structure $\mathcal{M}$, and a definable semigroup S, does there always exist a definable manifold structure on S that makes it a topological semigroup?

Note: If the answer is positive, in general such manifold structure will not be unique!

Example

Let $\mathcal{M}$ be o-minimal and X any definable set. Fix an element $0 \in X$ and consider $* : X \times X \to X$ such that $X * X = \{0\}$.
Any topology on X makes X a topological semigroup.

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