Introduction to Category Theory

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Categories

Definition

A *category* $\mathcal{C}$ consists of the following data:

- A collection $\mathrm{Ob}(\mathcal{C})$ of *objects*;
- For each pair of objects A, B, a collection $\hom_{\mathcal{C}}(A, B)$ whose elements are called *morphisms* or *arrows*;
- Composition: Given two morphisms $f : A \to B$ e $g : B \to C$, there exists a morphism $g \circ f : A \to C$;

Such that:

- Composition is associative;
- For each object A, there exists a morphism $1_A : A \to A$ called the *identity morphism* that acts as an identity for composition;

Notation:

$$\text{We write } A \in \mathcal{C} \text{ for } A \in \mathrm{Ob}(\mathcal{C})$$
$$\hom_{\mathcal{C}}(A, B),\ \hom(A, B) \text{ e } \mathcal{C}(A, B)$$

Examples

- **Set**, **Grp**, **Mon**, **Ring**, $\mathbf{Vect}_{\mathbb{R}}$, **Top**
- **N**
- Preorder / Partial order $(X, \leq)$
- $\mathbf{Mat}(\mathbb{R})$
- Monoid M / group G

Types of Morphisms

Definition

Let $\mathcal{C}$ be a category. A morphism $f : A \to B$ is said to be an *isomorphism* or *invertible*, if there exists a morphism $g : B \to A$ such that $gf = 1_A$ e $fg = 1_B$.
If that is the case, we say that the objects A and B are *isomorphic*, which we denote by $A \cong B$.

Note: if f is an isomorphism, then its inverse is unique, so we use the notation f^{-1}.

Definition

A group G is a category with only one object where every morphism is an isomorphism.

Note: A category where every morphism is an isomorphism is called a *groupoid*.

Monomorphisms

Proposition

Let X, Y be sets and $f : X \to Y$ a function. The following are equivalent:

- *f is injective;*
- *Given functions g, h:*

$$Z \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

  If $fg = fh$, then $g = h$;
- *There exists a function $w : Y \to X$ such that $wf = 1_X$.*

Definition

Let $\mathcal{C}$ be a category. A morphism $f : A \to B$ is a *monomorphism* or *mono* if, for any morphisms $g, h : C \rightrightarrows A$, the equality $fg = fh$ implies that $g = h$.

Epimorphisms

Proposition

Let X, Y be sets and $f : X \to Y$ a function. The following are equivalent:

- *f is surjective;*
- *Given two functions g, h:*

$$X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$$

  If $gf = hf$, then $g = h$;
- *There exists a function $w : Y \to X$ such that $fw = 1_Y$.*

Definition

Let $\mathcal{C}$ be a category. A morphism $f : A \to B$ is an *epimorphism* or *epic* if, for any morphisms $g, h : B \rightrightarrows C$, $gf = hf$ implies $g = h$.

Iso, Mono e Epi

Proposition

Let $\mathcal{C}$ be a category. If f is an isomorphism, then f is both a monomorphism and an epimorphism.

Note: f being a monomorphism + epimorphism does not imply that f is an isomorphism.
A category where this happens is called *balanced*.

Terminal and initial objects

Definition

Let $\mathcal{C}$ be a category. An object A is called:

- *terminal* if for every object B there exists exactly one morphism $B \to A$;
- *initial* if for every object B there exists exactly one morphism $A \to B$.

Proposition

Let $\mathcal{C}$ be a category. Then:

- *If $\mathcal{C}$ has a terminal object, then it is unique up to isomorphism;*
- *If $\mathcal{C}$ has an initial object, then it is unique up to isomorphism;*

Note: the terminal object is usually denoted by 1 and the initial object by 0 or $\emptyset$.

Opposite Category

Definition

Let $\mathcal{C}$ be a category. The *opposite category* of $\mathcal{C}$, denoted by $\mathcal{C}^{\mathrm{op}}$, is given by:

- The objects of $\mathcal{C}^{\mathrm{op}}$ are the objects of $\mathcal{C}$;
- The morphisms $\mathcal{C}^{\mathrm{op}}$ are the same as the morphisms in $\mathcal{C}$, but we 'flip' the direction. Formally:

$$\hom_{\mathcal{C}^{\mathrm{op}}}(A, B) := \hom_{\mathcal{C}}(B, A)$$

- Composition is given by $g \circ_{\mathrm{op}} f = f \circ g$;

Proposition

Let $\mathcal{C}$ be a category. Then:

- *An object A is terminal [initial] em $\mathcal{C}$ iff A is initial [terminal] in $\mathcal{C}^{\mathrm{op}}$;*
- *A morphism f is mono [epic] in $\mathcal{C}$ iff f is epic [mono] in $\mathcal{C}^{\mathrm{op}}$.*

Product of Categories

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We define $\mathcal{C} \times \mathcal{D}$ as being the category given by:

- The objects are pairs (A, B), where $A \in \mathcal{C}$ e $B \in \mathcal{D}$. In other words $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$;
- A morphism $(A, B) \to (A', B')$ is given by a pair (f, g) where $f : A \to A'$ is a morphism in $\mathcal{C}$ and $g : B \to B'$ is a morphism in $\mathcal{D}$;

Composition of two morphisms:

$$(A, B) \xrightarrow{(f,g)} (A', B') \xrightarrow{(f',g')} (A'', B'')$$

is defined as $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$.

Functors

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is given by:

- For each object $A \in \mathcal{C}$, an object $F(A)$ in $\mathcal{D}$;
- Given a morphism $f : A \to B$ em $\mathcal{C}$, a morphism $F(f)$ in $\mathcal{D}$:

$$\begin{array}{ccc} A & & F(A) \\ {\scriptstyle f}\big\downarrow & \overset{F}{\longmapsto} & \big\downarrow{\scriptstyle F(f)} \\ B & & F(B) \end{array}$$

Such that $F(1_A) = 1_{F(A)}$ for every object $A \in \mathcal{C}$, and $F(f \circ g) = F(f) \circ F(g)$ for all composable morphisms $A \xrightarrow{g} B \xrightarrow{f} C$ in $\mathcal{C}$.

Functors II

Definition

Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{K}$ be functors. We define the composition $G \circ F : \mathcal{C} \to \mathcal{K}$ to be the functor given by:

- For any $A \in \mathcal{C}$, then $(G \circ F)(A) = G(F(A))$;
- For any morphism f in $\mathcal{C}$, then $(G \circ F)(f) = G(F(f))$.

Definition

We define **Cat** as being the category whose objects are categories, and morphisms are functors, where composition is defined as above.

Definition

Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be isomorphic if they are isomorphic as objects in **Cat**. In other words, if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $GF = 1_{\mathcal{C}}$ e $FG = 1_{\mathcal{D}}$.

Full and Faithful

Note: Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. For each pair of objects $A, B \in \mathcal{C}$, the functor F induces a map

$$\begin{aligned} F : \hom_{\mathcal{C}}(A, B) &\to \hom_{\mathcal{D}}(F(A), F(B)) \\ f &\mapsto F(f) \end{aligned} \tag{1}$$

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be:

- *Full*, if the map (1) is surjective;
- *Faithful*, if the map (1) is injective.

Proposition

Let $F : \mathcal{C} \to D$ be a functor. Then F is an isomorphism iff it is bijective on objects, and fully faithful.

Functors III

Proposition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then:

1. *F preserves isomorphisms;*
2. *If F is fully faithful, then F reflects isomorphisms;*
3. *If F is faithful, then F reflects monomorphisms and epimorphisms.*

Corollary

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and A, B objects in $\mathcal{C}$. If $A \cong B$, then $F(A) \cong F(B)$, where the converse is true if, for example, F is fully faithful.

Contravariant Functors

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is given by:

- For each object $A \in \mathcal{C}$, an object $F(A)$ in $\mathcal{D}$;
- Given a morphism $f : A \to B$ em $\mathcal{C}$, a morphism $F(f)$ in $\mathcal{D}$:

$$\begin{array}{ccc} A & & F(A) \\ {\scriptstyle f}\big\downarrow & \overset{F}{\longmapsto} & \big\uparrow{\scriptstyle F(f)} \\ B & & F(B) \end{array}$$

Such that $F(1_A) = 1_{F(A)}$ for every object $A \in \mathcal{C}$, and $F(f \circ g) = F(g) \circ F(f)$ for all composable morphisms $A \xrightarrow{g} B \xrightarrow{f} C$ in $\mathcal{C}$.

Contravariant Functors II

Proposition

A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is the same thing as a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Proposition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is also a functor from $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

Note: As such, a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is the same thing as a functor $F : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.

Proposition

The assignment $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ and $F \mapsto F$ defines a functor $\mathbf{Cat} \to \mathbf{Cat}$.

Natural Transformations

Definition

Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors. A *natural transformation* $\alpha : F \Rightarrow G$ is given by, for each $C \in \mathcal{D}$, a morphism

$$\alpha_C : F(C) \to G(C)$$

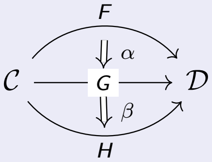
in $\mathcal{D}$, called the *components* of α in C, such that for each morphism $f : A \to B$ in $\mathcal{C}$, the following diagram called the *naturality square* of α commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\alpha_B]{} & G(B) \end{array}$$

Composition of Natural Transformations

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and consider the following functors and natural transformations:



We define the composite $\beta \circ \alpha : F \Rightarrow H$ as the natural transformation with components given by the composition of α_A followed by β_A:

$$(\beta \circ \alpha)_A : F(A) \xrightarrow{\alpha_A} G(A) \xrightarrow{\beta_A} H(A)$$

so $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ for each $A \in \mathcal{C}$.

Functor Categories

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We define $[\mathcal{C}, \mathcal{D}]$ as being the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms are natural transformations.

Definition

Two functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ are said to be *naturally isomorphic*, written as $F \cong G$, if they are isomorphic as objects in $[\mathcal{C}, \mathcal{D}]$, i.e. is there are natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow F$ such that $\alpha \circ \beta = 1_G$ and $\beta \circ \alpha = 1_F$. If this is the case, we say that α and β are *natural isomorphisms*.

Proposition

Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors and $\alpha : F \Rightarrow G$ a natural transformation. Then α is a natural isomorphism iff for each $A \in \mathcal{C}$, the component $\alpha_A : F(A) \to G(A)$ is an isomorphism in $\mathcal{D}$.

Equivalent Categories

Note: Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, we say that

$$F(A) \cong G(A), \text{ naturally in } A$$

to mean that F and G are naturally isomorphic.

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, denoted by $\mathcal{C} \simeq \mathcal{D}$, if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \cong 1_{\mathcal{D}}$ and $G \circ F \cong 1_{\mathcal{C}}$. If this is the case, then we say that F is an *equivalence* of categories.

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *essentially surjective on objects*, if for each $B \in \mathcal{D}$, there is some $A \in \mathcal{C}$ such that $F(A) \cong B$.

Proposition

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an equivalence iff F is fully faithful and essentially surjective on objects.

SubCategories

Definition

Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ is given by:

- A subset $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$ of objects;
- For each pair $A, B \in \mathrm{Ob}(\mathcal{D})$, a subset of morphisms $\hom_{\mathcal{D}}(A, B) \subseteq \hom_{\mathcal{C}}(A, B)$

Such that $\mathcal{D}$ is itself a category.

Definition

A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is said to be a *full subcategory* if $\hom_{\mathcal{D}}(A, B) = \hom_{\mathcal{C}}(A, B)$ for all $A, B \in \mathcal{D}$, i.e. if the inclusion functor $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ is full. (note that the inclusion functor is always faithful).

SubCategories II

Definition

A category $\mathcal{C}$ is said to be *skeletal*, if for any two objects $A, B \in \mathcal{C}$, if $A \cong B$ then $A = B$.
A *skeleton* of a category $\mathcal{C}$ is a skeletal subcategory whose inclusion functor is an equivalence.

Theorem

Every category has a skeleton. Furthermore, such skeleton is unique up to equivalence of categories.

Dualities

An equivalence $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{D}$ is called a *duality* between the categories $\mathcal{C}$ and $\mathcal{D}$.

Examples:

- Stone duality:

$$\{\text{Boolean Algebras}\}^{\mathrm{op}} \simeq \{\text{totally disconnected compact } T_2 \text{ spaces}\}$$

- Gelfand-Naimark duality:

$$\{\text{Commutative unital } C^*\text{-algebras}\}^{\mathrm{op}} \simeq \{\text{compact } T_2 \text{ spaces}\}$$

- In algebraic geometry, given a ACF k:

$$\{\text{Affine Varieties over } k\}^{\mathrm{op}} \simeq \{\text{f.g. } k\text{-algebras with no nontrivial nilpotents}\}$$

- Pontryagin Duality:

$$\{\text{Locally compact abelian top. Groups}\}^{\mathrm{op}} \simeq \{\text{Locally compact abelian top. Groups}\}$$